

## Curvilinear coordinates and reciprocal basis

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### 1.1 Motivation

Here I'd like to explore some ideas from [1] where curvilinear coordinates, manifolds, and the vector derivative are introduced.

### 1.2 Notation

For simplicity, let's consider the concrete example of a 2D manifold, a surface in an  $n$  dimensional vector space, parameterized by two variables

$$\mathbf{x} = \mathbf{x}(a, b) = \mathbf{x}(u^1, u^2). \quad (1.1)$$

Note that the indices here do not represent exponentiation. We can construct a basis for the manifold as

$$\mathbf{x}_i = \frac{\partial \mathbf{x}}{\partial u^i}. \quad (1.2)$$

On the manifold we can calculate a reciprocal basis  $\{\mathbf{x}^i\}$ , defined by requiring, at each point on the surface

$$\mathbf{x}^i \cdot \mathbf{x}_j = \delta^i_j. \quad (1.3)$$

Associated implicitly with this basis is a curvilinear coordinate representation defined by the projection operation

$$\mathbf{x} = x^i \mathbf{x}_i, \quad (1.4)$$

(sums over mixed indexes are implied). These coordinates can be calculated by taking dot products with the reciprocal frame vectors

$$\begin{aligned} \mathbf{x} \cdot \mathbf{x}^i &= x^j \mathbf{x}_j \cdot \mathbf{x}^i \\ &= x^j \delta_j^i \\ &= x^i. \end{aligned} \quad (1.5)$$

### 1.3 Examples

Let's pause for a couple examples that have interesting aspects.

#### Example 1.1 Circular coordinates on a disk

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Consider an infinite disk at height  $z_0$ , with the origin omitted, parameterized by circular coordinates as in fig. 1.1.

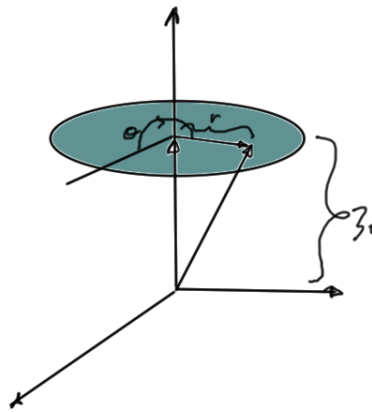


Figure 1.1: Plane with circular coordinates

Points on this surface are

$$\mathbf{x}(r, \theta) = (r \cos \theta, r \sin \theta, z_0). \quad (1.6)$$

The manifold basis vectors, defined by eq. (1.2) are

$$\begin{aligned} \mathbf{x}_r &= (\cos \theta, \sin \theta, 0) \\ \mathbf{x}_\theta &= r(-\sin \theta, \cos \theta, 0). \end{aligned} \quad (1.7)$$

By inspection, the reciprocal basis is

$$\begin{aligned} \mathbf{x}^r &= (\cos \theta, \sin \theta, 0) \\ \mathbf{x}^\theta &= \frac{1}{r}(-\sin \theta, \cos \theta, 0). \end{aligned} \quad (1.8)$$

The first thing to note here is that we cannot reach the points  $\mathbf{x}$  of eq. (1.6) by linear combination of these basis vectors. Instead these basis vectors only allow us to reach other points on the surface, when already there. For example we cannot actually write

$$\mathbf{x} = x^r \mathbf{x}_r + x^\theta \mathbf{x}_\theta, \quad (1.9)$$

unless  $z_0 = 0$ . This is why eq. (1.4) was described as a projective operation (and probably deserves an alternate notation). To recover the original parameterized form of the position vector on the surface, we require

$$\mathbf{x} = x^r \mathbf{x}_r + x^\theta \mathbf{x}_\theta + z_0 \hat{\mathbf{z}}. \quad (1.10)$$

The coordinates  $x^r, x^\theta$  follow by taking dot products

$$\begin{aligned} x^r &= \mathbf{x} \cdot \mathbf{x}^r \\ &= (r \cos \theta, r \sin \theta, z_0) \cdot (\cos \theta, \sin \theta, 0) \\ &= r (\cos^2 \theta + \sin^2 \theta) \\ &= r \end{aligned} \quad (1.11a)$$

$$\begin{aligned} x^\theta &= \mathbf{x} \cdot \mathbf{x}^\theta \\ &= (r \cos \theta, r \sin \theta, z_0) \cdot \frac{1}{r} (-\sin \theta, \cos \theta, 0) \\ &= 0. \end{aligned} \quad (1.11b)$$

Therefore, a point on the plane, relative to the origin of the plane, in this case, requires just one of the tangent plane basis vectors

$$\mathbf{x} = r \mathbf{x}_r. \quad (1.12)$$

### Example 1.2      Circumference of a circle

Now consider a circular perimeter, as illustrated in fig. 1.2, with the single variable parameterization

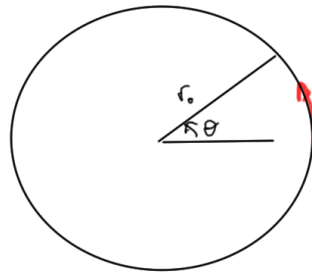


Figure 1.2: Circular perimeter

$$\mathbf{x} = r_0 (\cos \theta, \sin \theta). \quad (1.13)$$

Our tangent space basis is

$$\mathbf{x}_\theta = r_0 (-\sin \theta, \cos \theta), \quad (1.14)$$

with, by inspection, a reciprocal basis

$$\mathbf{x}^\theta = \frac{1}{r_0} (-\sin \theta, \cos \theta). \quad (1.15)$$

Here we have a curious condition, since the tangent space basis vector is perpendicular to the position vector for the points on the circular surface. So, should we attempt to calculate coordinates using eq. (1.4), we just get zero

$$\begin{aligned} x^\theta &= \mathbf{x} \cdot \mathbf{x}^\theta \\ &= r_0 (\cos \theta, \sin \theta) \cdot \frac{1}{r_0} (-\sin \theta, \cos \theta) \\ &= 0. \end{aligned} \quad (1.16)$$

It's perhaps notable that a coordinate representation using the tangent space basis is possible, but we need to utilize a complex geometry. Assuming

$$\mathbf{x} = x^\theta \mathbf{x}_\theta, \quad (1.17)$$

and writing  $i = \mathbf{e}_1 \mathbf{e}_2$  for the pseudoscalar, we can write

$$\begin{aligned} \mathbf{x} &= r_0 \mathbf{e}_1 e^{i\theta} \\ \mathbf{x}_\theta &= r_0 \mathbf{e}_2 e^{i\theta}, \end{aligned} \quad (1.18)$$

so that, by inversion, the  $\theta$  coordinate is

$$\begin{aligned} x^\theta &= \mathbf{x} \frac{1}{\mathbf{x}_\theta} \\ &= \left( r_0 \mathbf{e}_1 e^{i\theta} \right) \left( \frac{e^{-i\theta} \mathbf{e}_2}{r_0} \right) \\ &= i, \end{aligned} \quad (1.19)$$

or

$$\mathbf{x} = i \mathbf{x}_\theta. \quad (1.20)$$

### Example 1.3 Surface of a sphere

It is also clear that any parameterization that has radial symmetry will suffer the same issue. For example, for a radial surface in 3D with radius  $r_0$  we have

$$\begin{aligned}
\mathbf{x} &= r_0 (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \\
\mathbf{x}_\theta &= r_0 (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) \\
\mathbf{x}_\phi &= r_0 (-\sin \theta \sin \phi, \sin \theta \cos \phi, 0) \\
\mathbf{x}^\theta &= \frac{1}{r_0} (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) \\
\mathbf{x}^\phi &= \frac{1}{r_0 \sin \theta} (-\sin \phi, \cos \phi, 0).
\end{aligned} \tag{1.21}$$

The reciprocals here were computed using the mathematica `reciprocalFrameSphericalSurface.nb` notebook.

Do we have a bivector parameterization of the surface using the tangent space basis? Let's try

$$\mathbf{x} = x^\theta \mathbf{x}_\theta + x^\phi \mathbf{x}_\phi. \tag{1.22}$$

Wedging with  $\mathbf{x}_\theta$  and  $\mathbf{x}_\phi$ , and writing  $i = \mathbf{e}_1 \mathbf{e}_2$ , respectively yields

$$\begin{aligned}
x^\theta &= \mathbf{x} \wedge \mathbf{x}_\phi \frac{1}{\mathbf{x}_\theta \wedge \mathbf{x}_\phi} \\
&= -\mathbf{e}_1 \mathbf{e}_3 \cos \phi - \mathbf{e}_2 \mathbf{e}_3 \sin \phi \\
&= \mathbf{e}_{31} e^{i\phi}.
\end{aligned} \tag{1.23}$$

$$\begin{aligned}
x^\phi &= -\mathbf{x} \wedge \mathbf{x}_\theta \frac{1}{\mathbf{x}_\theta \wedge \mathbf{x}_\phi} \\
&= \mathbf{e}_1 \mathbf{e}_3 \cot \theta \sin \phi + i - \mathbf{e}_2 \mathbf{e}_3 \cot \theta \cos \phi \\
&= \mathbf{e}_2 \mathbf{e}_3 \cot \theta e^{i\phi} + i.
\end{aligned} \tag{1.24}$$

However, substitution back into eq. (1.22) shows either pair parameterizes the radial position vector

$$\mathbf{x} = x^\theta \mathbf{x}_\theta = x^\phi \mathbf{x}_\phi. \tag{1.25}$$

It is interesting that duality relationships seem to naturally arise attempting to describe points on a surface using the tangent space basis for that surface.

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## Bibliography

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- [1] A. Macdonald. *Vector and Geometric Calculus*. CreateSpace Independent Publishing Platform, 2012. 1.1