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Stokes theorem notes

Stokes law in wedge product form

2.1 A hodge podge of relations

The aim of these notes is to work through proofs of the following integral equations

- Gradient line integral.

$$\int_C (\nabla f) \cdot d\mathbf{r} = f|_{\partial C} \quad (2.1)$$

- Jacobian area determinants.

Change of variables for a double integral

$$dA = dxdy = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} dudv = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv \quad (2.2)$$

In Salus and Hille this is proved using Green's theorem, despite it seeming like the more basic operation. The greater than two dimensional cases are not proved at all.

- Green's theorem.

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint Pdx + Qdy \quad (2.3)$$

- Stokes theorem.

$$\iint (\nabla \times \mathbf{f}) \cdot \hat{\mathbf{n}} dxdy = \oint \mathbf{f} \cdot d\mathbf{r} \quad (2.4)$$

- Divergence theorem.

$$\iint \nabla \cdot \mathbf{f} dxdy = \oint \mathbf{f} \cdot \hat{\mathbf{n}} ds \quad (2.5)$$

$$\iiint_V \nabla \cdot \mathbf{f} \, dx \, dy \, dz = \iint_S \mathbf{f} \cdot \hat{\mathbf{n}} \, dA \quad (2.6)$$

$$\iiint_V \nabla \phi \, dV = \iint_S \hat{\mathbf{n}} \phi \, dA \quad (2.7)$$

$$\iiint_V \nabla \times \mathbf{f} \, dV = \iint_S \mathbf{f} \times \hat{\mathbf{n}} \, dA \quad (2.8)$$

In particular I had like to relate these to the geometrical concepts of Clifford algebra now that I know how to work with that in a differential and algebraic fashion for many sorts of problems. I am hoping that working through proofs of these basic identities will be enough that I can go on to the more general approaches in differential forms and the geometric calculus of Hestenes.

John Denker's [article on the magnetic field of a straight wire](#) gives a simple looking high level description of vector form of Stokes' theorem in its Clifford formulation

$$\int_S \nabla \wedge F = \int_{\partial S} F \quad (2.9)$$

This is simple enough looking, but there are some important details left out. In particular the grades do not match, so there must be some sort of implied projection or dot product operations too.

I had say this suffers from some of the things that I had trouble with in attempting to study differential forms.

The basic ideas of how to formulate the curve, surface, volume, ... of integration is not specified. How to do that in greater than three dimensions is not trivial seeming to me since none of the traditional methods of dotting with a normal will not work.

Knowing now about how subspaces can be expressed using blades is likely the key. The Clifford algebra ideas seem particularly suited to this as many of these ideas can be formulated independent of the calculus applications. One can learn the geometric and algebraic concepts first and then move on to the Calculus.

2.2 Gradient line integral

This is the easiest of the identities to prove. Introduction of a reciprocal frame $\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu$ also means that we can do in full generality with a possibly non-orthonormal basis of any dimension, and an arbitrary metric.

Write the gradient as normal

$$\nabla = \sum \gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma^\mu \partial_\mu$$

Here summation convention with implied sum over mixed upper and lower indices is employed.

Express the position vector along the curve as a parametrized path $\mathbf{r} = \mathbf{r}(\lambda) = \gamma_\mu x^\mu$, and use this to form the element of vector length along the path

$$d\mathbf{r} = \gamma_\mu \frac{dx^\mu}{d\lambda} d\lambda$$

Dotting the gradient and the path element we have

$$\begin{aligned}
 \nabla f \cdot d\mathbf{r} &= (\gamma^\mu \partial_\mu f) \cdot \left(\gamma_\nu \frac{dx^\nu}{d\lambda} \right) d\lambda \\
 &= \delta^\mu_\nu \frac{\partial f}{\partial x^\mu} \frac{dx^\nu}{d\lambda} d\lambda \\
 &= \sum \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\lambda} d\lambda \\
 &= \frac{df}{d\lambda} d\lambda
 \end{aligned} \tag{2.10}$$

Eq. (2.1) follows immediately, which we see to be really not much more than the chain rule.

Additionally this can be put into correspondence with eq. (2.9), with the observation that one can write the gradient of a scalar function as a wedge product by the fundamental definition of wedge in terms of grade selection. For blades A and B with grades a and b respectively, the wedge is

$$A \wedge B = \langle AB \rangle_{a+b} \tag{2.11}$$

Therefore for a scalar function f

$$\nabla \wedge f = \langle \nabla f \rangle_{1+0} = \nabla f \tag{2.12}$$

Putting this back together one has the desired result

$$\int_C (\nabla \wedge f) \cdot d\mathbf{r} = f|_{\partial C} \tag{2.13}$$

2.2.1 *Motivating the non-orthonormal form of the gradient*

An additional note about the derivation of this line integral result. Having done this with the gradient expressed for possibly non-orthonormal frames, shows that if played backwards, it provides a nice motivation for the general form of the gradient, in terms of a such a non-orthonormal basis. That is a lot more obvious a way to get at this result than my previous way of observing that the Euler-Lagrange equations when summed in vector form imply that this is the required form of the gradient.

2.3 *Jacobian area determinants*

Next in ease of proof is the Jacobian determinant. This actually comes largely for free since we can utilize the wedge product to express areas.

Introduce a two vector parametrization of the area as in fig. 2.1

$$\mathbf{r} = \gamma_i \phi^i(u, v)$$

Provided that the partials are not collinear at the point of interest, we can compute the area of the parallelogram spanned by these

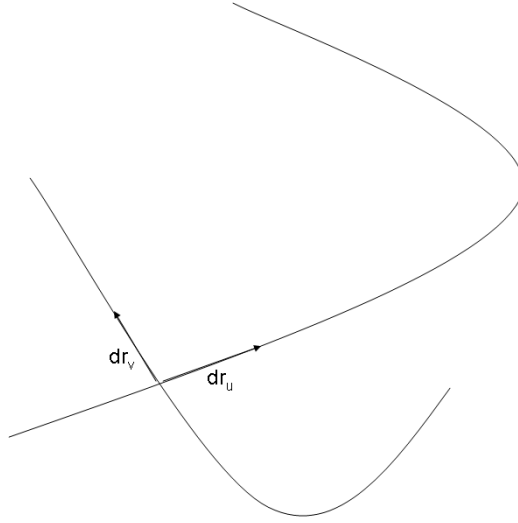


Figure 2.1: Plane parametrization

$$\begin{aligned}
 d\mathbf{A} &= \left(\frac{\partial \mathbf{r}}{\partial u} du \right) \wedge \left(\frac{\partial \mathbf{r}}{\partial v} dv \right) \\
 &= \left(\gamma_i \frac{\partial \phi^i}{\partial u} \right) \wedge \left(\gamma_j \frac{\partial \phi^j}{\partial v} \right) dudv \\
 &= \gamma_i \wedge \gamma_j \frac{\partial \phi^i}{\partial u} \frac{\partial \phi^j}{\partial v} dudv \\
 &= \sum_{i < j} \gamma_i \wedge \gamma_j \left(\frac{\partial \phi^i}{\partial u} \frac{\partial \phi^j}{\partial v} - \frac{\partial \phi^i}{\partial v} \frac{\partial \phi^j}{\partial u} \right) dudv \\
 &= \sum_{i < j} \gamma_i \wedge \gamma_j \frac{\partial(\phi^i, \phi^j)}{\partial(u, v)} dudv
 \end{aligned} \tag{2.14}$$

Here $d\mathbf{A}$ is a bivector area element, so in the purely two dimensional case, where this is constrained to a plane, the scalar area element is recovered by dividing by the plane unit pseudoscalar having the same orientation as this bivector.

One can also see how the same idea will be of use later in the Stokes' generalization of Green's theorem (considering a surface element small enough to be considered planar).

For now, considering just the 2D case we have, to divide through by the plane unit pseudoscalar $i = \mathbf{e}_1 \mathbf{e}_2$ produced by the product of two orthonormal vectors we want to calculate the product:

$$\begin{aligned}
\frac{1}{i} \gamma_1 \wedge \gamma_2 &= (\mathbf{e}_2 \wedge \mathbf{e}_1) \cdot (\gamma_1 \wedge \gamma_2) \\
&= \mathbf{e}_2 \cdot (\mathbf{e}_1 \cdot (\gamma_1 \wedge \gamma_2)) \\
&= \mathbf{e}_2 \cdot ((\mathbf{e}_1 \cdot \gamma_1)\gamma_2 - (\mathbf{e}_1 \cdot \gamma_2)\gamma_1) \\
&= (\mathbf{e}_1 \cdot \gamma_1)(\mathbf{e}_2 \cdot \gamma_2) - (\mathbf{e}_1 \cdot \gamma_2)(\mathbf{e}_2 \cdot \gamma_1) \\
&= \begin{vmatrix} \mathbf{e}_1 \cdot \gamma_1 & \mathbf{e}_1 \cdot \gamma_2 \\ \mathbf{e}_2 \cdot \gamma_1 & \mathbf{e}_2 \cdot \gamma_2 \end{vmatrix}
\end{aligned} \tag{2.15}$$

Thus the (scalar) area element is

$$dA = \begin{vmatrix} \mathbf{e}_1 \cdot \gamma_1 & \mathbf{e}_1 \cdot \gamma_2 \\ \mathbf{e}_2 \cdot \gamma_1 & \mathbf{e}_2 \cdot \gamma_2 \end{vmatrix} \frac{\partial(\phi^1, \phi^2)}{\partial(u, v)} du dv \tag{2.16}$$

This is a slightly more general form than we are used to seeing since the position vector parametrization was allowed to be expressed in terms of an arbitrary (possibly non-orthonormal) basis. Also observe that the coefficients in the determinant preceding the Jacobian are exactly those of the matrix of the linear transformation between the two sets of basis vectors.

2.3.1 *Orthonormal parametrization*

For the special (and usual) case of an orthonormal parametrization

$$\mathbf{r} = x(u, v)\mathbf{e}_1 + y(u, v)\mathbf{e}_2 \tag{2.17}$$

the product of determinants in eq. (2.16) takes the usual form

$$dx dy = \frac{\partial(x, y)}{\partial(u, v)} du dv. \tag{2.18}$$

Now the danger of an expression like eq. (2.18) is that the differential notation for the determinant makes it seem almost obvious. Now, if you understand the wedge product origin you can state that obviousness after a little bit of algebra. However, in a book like Salus and Hille (used for Calculus I-III in UofT Engineering) they can not even derive this two dimensional case til close to the end of the book, since they required Green's theorem to do so. I had say that in that case it is not really so obvious. The geometrical background just is not there.

Note that there are degrees of freedom to alter the sign given an arbitrary pseudoscalar. This illustrates why the absolute value of the Jacobian determinant is used in some circumstances. Less dodgy is to say the positive area element after change of variables in a specific region is produced by dividing out the pseudoscalar with the same orientation as the area element bivector.

It is also not too hard to see that this idea will also work for change of variables for volume and higher dimensional volume elements, after wedging N partials. We just have to divide by the spatial (or higher dimensional) pseudoscalar of the same orientation associated with the parametrization.

2.3.2 Surface area in higher dimensions

As well as being able to use these ideas to express scalar area and volume, or higher dimensional generalizations, this can be used to calculate surface area in any number of dimensions. For a two parameter vector parametrization of a surface $\mathbf{r}(u, v)$ we can write

$$A = \iint \frac{1}{I_2(u, v)} \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) dudv \quad (2.19)$$

Here $I_2(u, v)$ is the unit pseudoscalar for the tangent space of the surface at the point of interest with the orientation of the bivector $d\mathbf{A} = \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}$.

This is in fact equivalent to the familiar normal form in 3D expressed in terms of a cross product

$$A = \iint \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \cdot \hat{\mathbf{n}}(u, v) dudv \quad (2.20)$$

but the expression of eq. (2.19), holds for any number of dimensions $N \geq 2$. As with the wedge product form, we have a requirement that the parametrization is not degenerate at any point, so the 3D de-generalization of our requirement that $\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \neq 0$ on the region of the surface of interest means that for 3D we simply require $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \neq 0$.

A consequence of non-degeneracy for the region of the surface area being integrated means that the sign of the bivector cannot change sign, so we have equivalence with the concept of outwards normal to the surface by picking the tangent space unit pseudoscalar to have the same orientation as the bivector area element.

2.4 Green's theorem

2.4.1 Attempt to arrive at a more natural vector form for Green's theorem

It is pretty clear glancing at eq. (2.3), that the left hand side can likely be expressed as the curl of a vector. By curl here is meant the more natural bivector "curl", where we form the operator $\nabla \wedge$.

To get a feel for this operation, here is a dumb expansion of such a product, where an orthonormal basis for the plane is assumed. Introduce a vector

$$\mathbf{f} = P\mathbf{e}_1 + Q\mathbf{e}_2$$

then compute

$$\begin{aligned} \nabla \wedge \mathbf{f} &= (\mathbf{e}_1 \partial_1 + \mathbf{e}_2 \partial_2) \wedge (P\mathbf{e}_1 + Q\mathbf{e}_2) \\ &= (\mathbf{e}_1 \wedge \mathbf{e}_2) (\partial_1 Q - \partial_2 P) \\ &= i (\partial_1 Q - \partial_2 P) \end{aligned} \quad (2.21)$$

This allows for writing the scalar alternating form as a vector relation

$$\partial_1 Q - \partial_2 P = -i(\nabla \wedge \mathbf{f}). \quad (2.22)$$

Let us continue to put the Green's theorem area integral in complete vector form. Since the area element can be expressed in vector form, introduce a vector parametrization $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2$. The element of area expressed in terms of this parametrization is

$$dA = \frac{1}{i} \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) dudv$$

Re-assembling the scalar alternating eq. (2.22), we can put the area integral completely in vector form

$$\iint -i(\nabla \wedge \mathbf{f}) \frac{1}{i} \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) dudv = - \iint (\nabla \wedge \mathbf{f}) \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) dudv \quad (2.23)$$

Considering the total differential of the position vector, it makes sense to introduce vector differential elements to express this

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv = d\mathbf{u} + d\mathbf{v}$$

We can then rewrite eq. (2.23) once more in a slightly cleaner form, independent of the specific parametrization

$$- \iint (\nabla \wedge \mathbf{f}) \cdot (d\mathbf{u} \wedge d\mathbf{v}) = - \iint (\nabla \wedge \mathbf{f}) \cdot d\mathbf{A} \quad (2.24)$$

Here we see that it becomes natural to work with the oriented bivector area element $d\mathbf{A} = d\mathbf{u} \wedge d\mathbf{v}$.

Having arrived at what is likely the most natural vector form eq. (2.24) for the area integral.

We should be able to integrate this in its most general form, dropping references to the original x , and y coordinates. If this is the correct form, we should end up with a vector line integral around a path after doing so, and thus prove Green's theorem.

FIXME: thought this, but am having trouble. Will try from the loop integral instead.

2.4.2 Expanding the area integral in terms of an arbitrary parametrization

The integral expression of eq. (2.24) is a form that can be examined independent of the original planar Green's theorem motivation. Let us expand this picking an arbitrary parametrization for both the area element and the vector. There will also be no need for now to work with the original 2D vectors.

Given a two parameter vector parametrization of a surface, and a reciprocal frame representation of our curled vector:

$$\begin{aligned} \mathbf{r}(u, v) &= \gamma_i x^i(u, v) \\ \mathbf{f} &= \gamma^j f_j \end{aligned} \quad (2.25)$$

a bivector parallelogram surface element can be then be expressed as

$$\begin{aligned} d\mathbf{A} &= \left(\frac{\partial \mathbf{r}}{\partial u} du \right) \wedge \left(\frac{\partial \mathbf{r}}{\partial v} dv \right) \\ &= (\gamma_i \wedge \gamma_j) \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} dudv \end{aligned} \quad (2.26)$$

and our differential form is

$$\begin{aligned}
(\nabla \wedge \mathbf{f}) \cdot d\mathbf{A} &= (\gamma^i \wedge \gamma^j) \cdot (\gamma_k \wedge \gamma_m) \frac{\partial f_j}{\partial x^i} \frac{\partial x^k}{\partial u} \frac{\partial x^m}{\partial v} dudv \\
&= (\delta_m^i \delta_k^j - \delta_k^i \delta_m^j) \frac{\partial f_j}{\partial x^i} \frac{\partial x^k}{\partial u} \frac{\partial x^m}{\partial v} dudv \\
&= \frac{\partial f_j}{\partial x^i} \left(\frac{\partial x^j}{\partial u} \frac{\partial x^i}{\partial v} - \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} \right) dudv \\
&= -\frac{\partial f_j}{\partial x^i} \frac{\partial(x^i, x^j)}{\partial(u, v)} dudv \\
&= -\sum_{i < j} \left(\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} \right) \frac{\partial(x^i, x^j)}{\partial(u, v)} dudv
\end{aligned} \tag{2.27}$$

The trailing differential form here is just the Jacobian form for change of variables, so we have

$$\begin{aligned}
-(\nabla \wedge \mathbf{f}) \cdot d\mathbf{A} &= \sum_{i < j} \left(\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} \right) \frac{\partial(x^i, x^j)}{\partial(u, v)} dudv \\
&= \sum_{i < j} \left(\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} \right) dx^i dx^j
\end{aligned} \tag{2.28}$$

A result that is independent of dimension or any particular parametrization of the area.

2.4.3 Calculating the line integral

The expectation is that calculation of the line integral

$$I = \oint \mathbf{f} \cdot d\mathbf{r}, \tag{2.29}$$

around any loop in a plane will match eq. (2.28). This can be verified with direct calculation.

FIXME: insert picture.

Again parametrizing the points around the loop with a vector $\mathbf{r} = \mathbf{r}(u, v)$ the integral can be split into four parts

$$\begin{aligned}
I_1 &= \int_{u=u_0}^{u_1} \mathbf{f}(\mathbf{r}(u, v_0)) \cdot \frac{\partial \mathbf{r}(u, v_0)}{\partial u} du \\
I_2 &= \int_{v=v_0}^{v_1} \mathbf{f}(\mathbf{r}(u_1, v)) \cdot \frac{\partial \mathbf{r}(u_1, v)}{\partial v} dv \\
I_3 &= - \int_{u=u_0}^{u_1} \mathbf{f}(\mathbf{r}(u, v_1)) \cdot \frac{\partial \mathbf{r}(u, v_1)}{\partial u} du \\
I_4 &= - \int_{v=v_0}^{v_1} \mathbf{f}(\mathbf{r}(u_0, v)) \cdot \frac{\partial \mathbf{r}(u_0, v)}{\partial v} dv
\end{aligned} \tag{2.30}$$

Summing these we have

$$\begin{aligned}
I &= I_1 + I_3 + I_2 + I_4 \\
&= \int_{u=u_0}^{u_1} du \left(\mathbf{f}(\mathbf{r}(u, v_0)) \cdot \frac{\partial \mathbf{r}(u, v_0)}{\partial u} - \mathbf{f}(\mathbf{r}(u, v_1)) \cdot \frac{\partial \mathbf{r}(u, v_1)}{\partial u} \right) \\
&\quad + \int_{v=v_0}^{v_1} dv \left(\mathbf{f}(\mathbf{r}(u_1, v)) \cdot \frac{\partial \mathbf{r}(u_1, v)}{\partial v} - \mathbf{f}(\mathbf{r}(u_0, v)) \cdot \frac{\partial \mathbf{r}(u_0, v)}{\partial v} \right)
\end{aligned} \tag{2.31}$$

Writing out the vectors in components, utilizing reciprocal frames as in the area integral, we have $\mathbf{f} = \gamma^i f_i$, and $\mathbf{r} = \gamma_i x^i$ the dot products can be expanded and the sums pulled out of the integral

$$\begin{aligned}
I &= \sum \int_{u=u_0}^{u_1} du \left(f_i(\mathbf{r}(u, v_0)) \frac{\partial x^i(u, v_0)}{\partial u} - f_i(\mathbf{r}(u, v_1)) \frac{\partial x^i(u, v_1)}{\partial u} \right) \\
&\quad + \sum \int_{v=v_0}^{v_1} dv \left(f_i(\mathbf{r}(u_1, v)) \frac{\partial x^i(u_1, v)}{\partial v} - f_i(\mathbf{r}(u_0, v)) \frac{\partial x^i(u_0, v)}{\partial v} \right)
\end{aligned} \tag{2.32}$$

The difference of functions here can be written as the integral of partials over the $[u_0, u_1]$ or $[v_0, v_1]$ ranges. This can be more obvious if one temporarily introduces helper functions of one variable describing the difference (FIXME: do so like on paper notes).

Such an integration gives

$$\begin{aligned}
I &= \sum \int_{u=u_0}^{u_1} du \left(f_i(x^j(u, v_0)) \frac{\partial x^i(u, v_0)}{\partial u} - f_i(x^j(u, v_1)) \frac{\partial x^i(u, v_1)}{\partial u} \right) \\
&\quad + \sum \int_{v=v_0}^{v_1} dv \left(f_i(x^j(u_1, v)) \frac{\partial x^i(u_1, v)}{\partial v} - f_i(x^j(u_0, v)) \frac{\partial x^i(u_0, v)}{\partial v} \right) \\
&= \sum - \int_{u=u_0}^{u_1} du \int_{v=v_0}^{v_1} dv \frac{\partial}{\partial v} \left(f_i(x^j(u, v)) \frac{\partial x^i(u, v)}{\partial u} \right) \\
&\quad + \sum \int_{v=v_0}^{v_1} dv \int_{u=u_0}^{u_1} du \frac{\partial}{\partial u} \left(f_i(x^j(u, v)) \frac{\partial x^i(u, v)}{\partial v} \right) \\
&= \sum \iint \left(\frac{\partial}{\partial u} \left(f_i \frac{\partial x^i}{\partial v} \right) - \frac{\partial}{\partial v} \left(f_i \frac{\partial x^i}{\partial u} \right) \right) dudv \\
&= \sum \iint \left(\frac{\partial f_i}{\partial u} \frac{\partial x^i}{\partial v} + f_i \frac{\partial}{\partial u} \frac{\partial x^i}{\partial v} - \frac{\partial f_i}{\partial v} \frac{\partial x^i}{\partial u} - f_i \frac{\partial}{\partial v} \frac{\partial x^i}{\partial u} \right) dudv \\
&= \sum \iint \left(\frac{\partial f_i}{\partial u} \frac{\partial x^i}{\partial v} - \frac{\partial f_i}{\partial v} \frac{\partial x^i}{\partial u} \right) dudv
\end{aligned} \tag{2.33}$$

Sufficient continuity in the coordinates x^i has been assumed here for mixed partial equality. Expanding out the partials with respect to u and v in terms of the coordinates one has

$$\begin{aligned}
I &= \sum \iint \frac{\partial f_i}{\partial x^j} \left(\frac{\partial x^j}{\partial u} \frac{\partial x^i}{\partial v} - \frac{\partial x^j}{\partial v} \frac{\partial x^i}{\partial u} \right) dudv \\
&= - \sum \iint \frac{\partial f_i}{\partial x^j} \frac{\partial(x^i, x^j)}{\partial(u, v)} dudv
\end{aligned} \tag{2.34}$$

Summing over $i < j$ and $j > i$, with a switch of variables we have

$$I = \sum_{i < j} \iint \left(\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} \right) \frac{\partial(x^i, x^j)}{\partial(u, v)} dudv \quad (2.35)$$

which equals the area integral of eq. (2.28). We have therefore proved a hybrid Green's-like and Stokes-like theorem

$$\iint (\nabla \wedge \mathbf{f}) \cdot d\mathbf{A} = \oint \mathbf{f} \cdot d\mathbf{r} \quad (2.36)$$

Like the \mathbb{R}^2 Green's result this applies to a looping path integral in a plane, but this form is valid for $\mathbf{f} \in \mathbb{R}^N$ as well. In particular, like Stokes' law this this applies to \mathbb{R}^3 .

I set out only to prove Green's theorem, but basically got the general proof without much extra work (using i, j instead of 1, 2) once the area integral was expressed as in terms of the wedge curl.

Note carefully that there is a difference in the direction of the path integral compared to the cross product form of Stokes' law since the squared bivector on the LHS introduces a negation.

To generalize this to a non-planar surface, the usual additional arguments to express a general surface as a triangularized set of differential plane elements are required, with summation canceling opposing interior contributions, is required to complete the proof. The interesting (to me) part is the plane to line integral Stokes law equation has been expressed in its \mathbb{R}^N generality, and without omission of the important loop orientation, and area sense.

2.4.4 Application to formulate Stokes law for a plane loop

We can obtain the \mathbb{R}^3 cross product form of Stokes law (assuming the triangularization generalization has been done) with some basic algebraic manipulations.

Let $\hat{\mathbf{n}}d\mathbf{A} = idA$, where $i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the \mathbb{R}^3 pseudoscalar. Inserting back into the differential form of the area integral of eq. (2.36) we have

$$\begin{aligned} (\nabla \wedge \mathbf{f}) \cdot d\mathbf{A} &= \langle (\nabla \wedge \mathbf{f}) \hat{\mathbf{n}}idA \rangle \\ &= \langle i(\nabla \times \mathbf{f}) \hat{\mathbf{n}}idA \rangle \\ &= -\langle (\nabla \times \mathbf{f}) \hat{\mathbf{n}} \rangle dA \\ &= -(\nabla \times \mathbf{f}) \cdot \hat{\mathbf{n}}dA \end{aligned} \quad (2.37)$$

This recovers the cross product form of Stokes law

$$\iint (\nabla \times \mathbf{f}) \cdot \hat{\mathbf{n}}dA = - \oint \mathbf{f} \cdot d\mathbf{r} = \oint \mathbf{f} \cdot d\mathbf{r}. \quad (2.38)$$

Note that the surface here does not have to have any notion of outwards facing normal (this makes no sense for a plane for example), as is usually used in the description of the \mathbb{R}^3 vector form of Stokes' law. That means some care is required in the definition of the unit normal $\hat{\mathbf{n}}$. There is however an orientation for this vector, and that is fixed by the pseudoscalar. Supposing that one picks $\hat{\mathbf{n}}$ such that $\hat{\mathbf{n}}dA \frac{1}{i} = dA > 0$. With such a selection, and $d\mathbf{A} = \hat{\mathbf{u}}\hat{\mathbf{v}}dA$, the triplet of vectors is oriented such that $\hat{\mathbf{n}}\hat{\mathbf{u}}\hat{\mathbf{v}} = i$. There is no notion of handedness required, which is a very \mathbb{R}^3 concept, despite having a notion of explicitly oriented vectors.

2.4.5 Volume integral to Area integral

Having calculated the scalar and vector variants of eq. (2.9), doing the same for the bivector case makes sense, especially since we need such a result for electromagnetism.

The aim will be to calculate

$$\iiint (\nabla \wedge F) \cdot d\mathbf{V} = \pm \iint F \cdot d\mathbf{A} \quad (2.39)$$

where the relative orientation of the area elements has yet to be determined.

Volume integral part

Expanding the LHS using

$$\begin{aligned} F &= \frac{1}{2} F_{ij} \gamma^{ij} \\ \gamma^{ij} &= \gamma^i \wedge \gamma^j \\ \nabla &= \gamma^i \partial_i \\ d\mathbf{V} &= (\partial_u \mathbf{r} \wedge \partial_v \mathbf{r} \wedge \partial_w \mathbf{r}) du dv dw \\ \mathbf{r} &= x^i \gamma^i \end{aligned} \quad (2.40)$$

where the volume is a three variable (u, v, w) parametrized parallelepiped volume element, we have

$$(\nabla \wedge F) \cdot d\mathbf{V} = \frac{1}{2} \gamma^{ijk} \cdot \gamma_{mnl} \partial_i F_{jk} \partial_u x^m \partial_v x^n \partial_w x^l du dv dw \quad (2.41)$$

Expanding the dot product term we have

$$\begin{aligned} \gamma^{ijk} \cdot \gamma_{mnl} &= (((\gamma^i \wedge \gamma^j \wedge \gamma^k) \cdot \gamma_m) \cdot \gamma_n) \cdot \gamma_l \\ &= (\gamma^{ij} \delta^k_m - \gamma^{ik} \delta^j_m + \gamma^{jk} \delta^i_m) \cdot \gamma_{nl} \\ &= \delta_l^i \delta_n^j \delta_m^k - \delta_l^j \delta_n^i \delta_m^k \\ &\quad - \delta_l^i \delta_n^k \delta_m^j + \delta_l^k \delta_n^i \delta_m^j \\ &\quad + \delta_l^j \delta_n^k \delta_m^i - \delta_l^k \delta_n^j \delta_m^i \end{aligned} \quad (2.42)$$

for short, write $\partial_{uvw}^{ijk} = \partial_u x^i \partial_v x^j \partial_w x^k$. Expanding the deltas in $\gamma^{ijk} \cdot \gamma_{mnl} \partial_{uvw}^{mnl}$ we have

$$\begin{aligned} &\delta_l^i \delta_n^j \delta_m^k \partial_{uvw}^{mnl} - \delta_l^j \delta_n^i \delta_m^k \partial_{uvw}^{mnl} - \delta_l^i \delta_n^k \delta_m^j \partial_{uvw}^{mnl} \\ &+ \delta_l^k \delta_n^i \delta_m^j \partial_{uvw}^{mnl} + \delta_l^j \delta_n^k \delta_m^i \partial_{uvw}^{mnl} - \delta_l^k \delta_n^j \delta_m^i \partial_{uvw}^{mnl} \end{aligned} \quad (2.43)$$

$$\begin{aligned} &= \partial_{uvw}^{kji} - \partial_{uvw}^{kij} - \partial_{uvw}^{jki} + \partial_{uvw}^{jik} + \partial_{uvw}^{ikj} - \partial_{uvw}^{ijk} \\ &\gamma^{ijk} \cdot \gamma_{mnl} \partial_{uvw}^{mnl} = \sum \epsilon_{ijk} \partial_{uvw}^{ijk} \\ &= \frac{\partial(x^i, x^j, x^k)}{\partial(u, v, w)} \end{aligned} \quad (2.44)$$

Therefore the final coordinate expression for the volume differential form is

$$(\nabla \wedge F) \cdot d\mathbf{V} = -\frac{1}{2} \partial_i F_{jk} \frac{\partial(x^i, x^j, x^k)}{\partial(u, v, w)} du dv dw. \tag{2.45}$$

Area integral part

We next want to compare to an oriented area dot product

$$\oiint F \cdot d\mathbf{A}$$

What is meant by an oriented area element in such an integral? I will try this calculation with loops drawn counterclockwise on each face of a parallelepiped, such that the adjacent loops “cancel out”. This eliminates a requirement for an outward normal concept which is only useful in \mathbb{R}^3 . This is still a very geometric concept, and a good mathematical description independent of pictures is required to get any further than a third degree volume subspace. In fact, to formulate this description, and the results below, I have drawn arrows on a cube, and labeled these arrows $\pm 1, \pm 2, \pm 3$ to enumerate them (also labeling the sides Front, Left, Right, Bottom, Top, and Posterior).

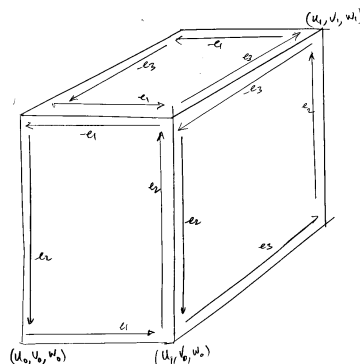


Figure 2.2: Oriented parallelepiped surfaces

With such a labeling we have the following table of paired area elements

- Front ($w = w_0$). Posterior ($w = w_1$).

$$\begin{aligned}
d\mathbf{A}_F &= \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) dudv \\
d\mathbf{A}_P &= \left(-\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) dudv
\end{aligned} \tag{2.46}$$

- Left ($u = u_0$). Right ($u = u_1$).

$$\begin{aligned}
d\mathbf{A}_L &= \left(-\frac{\partial \mathbf{r}}{\partial w} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) dwdv \\
d\mathbf{A}_R &= \left(\frac{\partial \mathbf{r}}{\partial w} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) dwdv
\end{aligned} \tag{2.47}$$

- Top ($v = v_1$). Bottom ($v = v_0$).

$$\begin{aligned}
d\mathbf{A}_T &= \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial w} \right) dudw \\
d\mathbf{A}_B &= \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \left(-\frac{\partial \mathbf{r}}{\partial w} \right) \right) dudw
\end{aligned} \tag{2.48}$$

With this particular enumeration of the oriented areas this geometrically described is a “left handed” triple $\left\{ \frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial w} \right\}$.

Summing the differential forms, writing our partial wedges in short like so $\mathbf{r}_{uv} = \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}$, we have

$$\begin{aligned}
\sum F \cdot d\mathbf{A} &= (F \cdot \mathbf{r}_{uv}|_{w=w_0} - F \cdot \mathbf{r}_{uv}|_{w=w_1}) dudv \\
&\quad + (F \cdot \mathbf{r}_{wv}|_{u=u_1} - F \cdot \mathbf{r}_{wv}|_{u=u_0}) dwdv \\
&\quad + (F \cdot \mathbf{r}_{uw}|_{v=v_1} - F \cdot \mathbf{r}_{uw}|_{v=v_0}) dudw \\
\Rightarrow \oiint F \cdot d\mathbf{A} &= \iint dudv \int_{w=w_0}^{w_1} -\frac{\partial}{\partial w} (F \cdot \mathbf{r}_{uv}) dw \\
&\quad + \iint dwdv \int_{u=u_0}^{u_1} \frac{\partial}{\partial u} (F \cdot \mathbf{r}_{wv}) du \\
&\quad + \iint dudw \int_{v=v_0}^{v_1} \frac{\partial}{\partial v} (F \cdot \mathbf{r}_{uw}) dv \\
&= \iiint dudvdw \left(\frac{\partial F}{\partial w} \cdot \mathbf{r}_{vu} + \frac{\partial F}{\partial u} \cdot \mathbf{r}_{wv} + \frac{\partial F}{\partial v} \cdot \mathbf{r}_{uw} \right) \\
&\quad + \iiint dudvdw F \cdot \left(\frac{\partial \mathbf{r}_{vu}}{\partial w} + \frac{\partial \mathbf{r}_{wv}}{\partial u} + \frac{\partial \mathbf{r}_{uw}}{\partial v} \right)
\end{aligned} \tag{2.50}$$

The last three terms here are expected to contribute zero. Picking one to start

$$\begin{aligned}
\frac{\partial \mathbf{r}_{vu}}{\partial w} &= \gamma^{ij} \frac{\partial}{\partial w} \left(\frac{\partial x^i}{\partial v} \frac{\partial x^j}{\partial u} \right) \\
&= \gamma^{ij} \left(\frac{\partial^2 x^i}{\partial w \partial v} \frac{\partial x^j}{\partial u} + \frac{\partial x^i}{\partial v} \frac{\partial^2 x^j}{\partial w \partial u} \right)
\end{aligned} \tag{2.51}$$

and summing this and the rest

$$\begin{aligned}
&\frac{\partial \mathbf{r}_{vu}}{\partial w} + \frac{\partial \mathbf{r}_{wv}}{\partial u} + \frac{\partial \mathbf{r}_{uw}}{\partial v} \\
&= \gamma^{ij} \left(\frac{\partial^2 x^i}{\partial w \partial v} \frac{\partial x^j}{\partial u} + \frac{\partial x^i}{\partial v} \frac{\partial^2 x^j}{\partial w \partial u} + \frac{\partial^2 x^i}{\partial u \partial w} \frac{\partial x^j}{\partial v} + \frac{\partial x^i}{\partial w} \frac{\partial^2 x^j}{\partial u \partial v} + \frac{\partial^2 x^i}{\partial v \partial u} \frac{\partial x^j}{\partial w} + \frac{\partial x^i}{\partial u} \frac{\partial^2 x^j}{\partial v \partial w} \right) \\
&= \gamma^{ij} \left(\left(\frac{\partial^2 x^i}{\partial w \partial v} \frac{\partial x^j}{\partial u} - \frac{\partial x^j}{\partial u} \frac{\partial^2 x^i}{\partial v \partial w} \right) + \left(\frac{\partial x^i}{\partial v} \frac{\partial^2 x^j}{\partial w \partial u} - \frac{\partial^2 x^j}{\partial u \partial w} \frac{\partial x^i}{\partial v} \right) + \left(\frac{\partial x^i}{\partial w} \frac{\partial^2 x^j}{\partial u \partial v} - \frac{\partial^2 x^j}{\partial v \partial u} \frac{\partial x^i}{\partial w} \right) \right) \\
&= 0,
\end{aligned} \tag{2.52}$$

yields the expected zero, independent of F . This leaves our area integral as follows

$$\Rightarrow \oiint F \cdot d\mathbf{A} = - \iiint dudvdw \left(\frac{\partial F}{\partial w} \cdot \mathbf{r}_{uv} + \frac{\partial F}{\partial u} \cdot \mathbf{r}_{vw} + \frac{\partial F}{\partial v} \cdot \mathbf{r}_{wu} \right) \tag{2.53}$$

To evaluate the sum to match with the volume integral we expand as in

$$\begin{aligned}
\frac{\partial F}{\partial w} \cdot \mathbf{r}_{uv} &= \frac{1}{2} \frac{\partial F_{ij}}{\partial w} (\gamma^{ij} \cdot \gamma_{mn}) \partial_u x^m \partial_v x^n \\
&= \frac{1}{2} \partial_w x^k \partial_k F_{ij} (\delta_n^i \delta_m^j - \delta_n^j \delta_m^i) \partial_u x^m \partial_v x^n \\
&= \frac{1}{2} \partial_k F_{ij} (\delta_n^i \delta_m^j - \delta_n^j \delta_m^i) \partial_u x^m \partial_v x^n \partial_w x^k \\
&= \frac{1}{2} \partial_k F_{ij} (\delta_n^i \delta_m^j - \delta_n^j \delta_m^i) \partial_{uvw}^{mnk}
\end{aligned} \tag{2.54}$$

Summing the dot products in eq. (2.53) we have

$$\begin{aligned}
&(\delta_n^i \delta_m^j - \delta_n^j \delta_m^i) (\partial_u x^m \partial_v x^n \partial_w x^k + \partial_v x^m \partial_w x^n \partial_u x^k + \partial_w x^m \partial_u x^n \partial_v x^k) \\
&= \delta_n^i \delta_m^j \partial_{uvw}^{mnk} + \delta_n^i \delta_m^j \partial_{uvw}^{kmn} + \delta_n^i \delta_m^j \partial_{uvw}^{nkm} - \delta_n^j \delta_m^i \partial_{uvw}^{mnk} - \delta_n^j \delta_m^i \partial_{uvw}^{kmn} - \delta_n^j \delta_m^i \partial_{uvw}^{nkm} \\
&= \partial_{uvw}^{jik} + \partial_{uvw}^{kji} + \partial_{uvw}^{ikj} - \partial_{uvw}^{ijk} - \partial_{uvw}^{kij} - \partial_{uvw}^{jki} \\
&= - \sum \epsilon_{ijk} \partial_{uvw}^{ijk} \\
&= - \frac{\partial(x^i, x^j, x^k)}{\partial(u, v, w)}
\end{aligned} \tag{2.55}$$

Finally we can put the area integral back together

$$\oiint F \cdot d\mathbf{A} = \iiint dudvdw \frac{1}{2} \partial_k F_{ij} \frac{\partial(x^i, x^j, x^k)}{\partial(u, v, w)} \tag{2.56}$$

A comparison to the volume eq. (2.45) shows a factor of -1 difference, which completes the proof and fixes the orientation of the surface area elements

$$\iiint (\nabla \wedge F) \cdot d\mathbf{V} = \oint F \cdot d\mathbf{A} \quad (2.57)$$

2.5 Divergence theorem

The divergence theorem results follow directly from the Stokes variants after duality transformations. Let us summarize all the Stokes equations proved so far to start

- $f \in \mathbb{R}^1$

$$\int_C (\nabla \wedge f) \cdot d\mathbf{r} = f|_{\partial C} \quad (2.58)$$

- $f \in \mathbb{R}^N$

$$\iint (\nabla \wedge f) \cdot d\mathbf{A} = \oint f \cdot d\mathbf{r} \quad (2.59)$$

It was demonstrated that Green's eq. (2.3) and the cross product Stokes eq. (2.4) results are the \mathbb{R}^2 and \mathbb{R}^3 special cases of this respectively.

- $f \in \wedge^2 \mathbb{R}^N$

$$\iiint (\nabla \wedge f) \cdot d\mathbf{V} = \oint f \cdot d\mathbf{A} \quad (2.60)$$

2.5.1 Two variable divergence (Gauss's law)

For the \mathbb{R}^2 divergence result we set $f = Ig$, for vectors $f, g \in \mathbb{R}^2$. Calculating the area differential form of eq. (2.59) we have

$$\begin{aligned} (\nabla \wedge f) \cdot d\mathbf{A} &= (\nabla \wedge (Ig)) \cdot d\mathbf{A} \\ &= \langle \nabla Ig \rangle_2 \cdot d\mathbf{A} \\ &= -\langle I\nabla g \rangle_2 \cdot d\mathbf{A} \\ &= -\langle I(\nabla \cdot g + \nabla \wedge g) \rangle_2 \cdot d\mathbf{A} \\ &= -I(\nabla \cdot g) \cdot d\mathbf{A} \\ &= -(\nabla \cdot g)Id\mathbf{A} \end{aligned} \quad (2.61)$$

Expanding the line integral side of the equation we have

$$\begin{aligned} f \cdot d\mathbf{r} &= (Ig) \cdot d\mathbf{r} \\ &= \langle Ig d\mathbf{r} \rangle \\ &= -\langle g Id\mathbf{r} \rangle \\ &= -g \cdot (Id\mathbf{r}) \end{aligned} \quad (2.62)$$

Therefore,

$$\iint (\nabla \cdot g) Id\mathbf{A} = \oint g \cdot (Id\mathbf{r}) \quad (2.63)$$

Letting $dA = Id\mathbf{A}$, and $\hat{\mathbf{n}}ds = I \cdot d\mathbf{r}$, we have the usual form for two variable Gauss's law:

$$\iint (\nabla \cdot g)dA = \oint g \cdot \hat{\mathbf{n}}ds. \quad (2.64)$$

Normally, the line integral side of this equation is expressed in terms of an outwards normal. A specific orientation for $\hat{\mathbf{n}}$ is also implied here, but depends on what bivector is used for the pseudoscalar I and the metric for the space, so for generality this is left implicitly defined as above.

2.5.2 Two variable divergence (Gauss's law)

For a bivector $F \in \wedge^2 \mathbb{R}^3$, we introduce a duality defined vector $f = IF$ (now I is an \mathbb{R}^3 pseudoscalar).

Expanding the two parts of the equation we have

$$\begin{aligned} (\nabla \wedge F) \cdot d\mathbf{V} &= (\nabla \wedge (If)) \cdot d\mathbf{V} \\ &= \langle \nabla If \rangle_3 \cdot d\mathbf{V} \\ &= \langle I \nabla f \rangle_3 \cdot d\mathbf{V} \\ &= \langle I(\nabla \cdot f + \nabla \wedge f) \rangle_3 \cdot d\mathbf{V} \\ &= I(\nabla \cdot f) \cdot d\mathbf{V} \\ &= (\nabla \cdot f)I \cdot d\mathbf{V} \\ &= (\nabla \cdot f)(Id\mathbf{V}) \end{aligned} \quad (2.65)$$

$$\begin{aligned} F \cdot d\mathbf{A} &= (If) \cdot d\mathbf{A} \\ &= \langle Ifd\mathbf{A} \rangle \\ &= \langle f(Id\mathbf{A}) \rangle \\ &= f \cdot (I \cdot d\mathbf{A}) \end{aligned} \quad (2.66)$$

With $dV = I \cdot d\mathbf{V}$, and $\hat{\mathbf{n}}d\sigma = I \cdot d\mathbf{A}$ we have the \mathbb{R}^3 divergence equation

$$\iiint (\nabla \cdot f)dV = \oint f \cdot \hat{\mathbf{n}}d\sigma \quad (2.67)$$

2.5.3 General divergence equation

Now, if one assumes the correctness of the Stokes result for a k -blade T (proven here only for the scalar, vector, and bivector case) has the form

$$\int (\nabla \wedge T) \cdot d^k \mathbf{x} = \int T \cdot d^{k-1} \mathbf{x} \quad (2.68)$$

then the divergence result for vector $\mathbf{t} = IT$ also follows without much more work than the two specific cases above

$$\begin{aligned}
(\nabla \wedge T) \cdot d^k \mathbf{x} &= \left\langle \nabla \frac{1}{I} \mathbf{t} \right\rangle_k \cdot d^k \mathbf{x} \\
&= (-1)^{k-1} \left\langle \frac{1}{I} \nabla \mathbf{t} \right\rangle_k \cdot d^k \mathbf{x} \\
&= (-1)^{k-1} \frac{1}{I} (\nabla \cdot \mathbf{t}) \cdot d^k \mathbf{x}
\end{aligned} \tag{2.69}$$

$$\begin{aligned}
T \cdot d^{k-1} \mathbf{x} &= \left\langle \frac{1}{I} \mathbf{t} d^{k-1} \mathbf{x} \right\rangle \\
&= (-1)^{k-1} \mathbf{t} \cdot \left(\frac{1}{I} d^{k-1} \mathbf{x} \right)
\end{aligned} \tag{2.70}$$

therefore, with $d^k x = I \cdot d^k \mathbf{x}$, and $\hat{\mathbf{n}} d^{k-1} x = I \cdot d^{k-1} \mathbf{x}$ we have

$$\int (\nabla \cdot \mathbf{t}) d^k x = \int \mathbf{t} \cdot \hat{\mathbf{n}} d^{k-1} x \tag{2.71}$$

Here $\hat{\mathbf{n}}$ is normal to all of the vectors in the span of the boundary surface ($\hat{\mathbf{n}} \wedge d^{k-1} \mathbf{x} \neq 0$). There are unspecified subtleties in eq. (2.71) since the orientation of the boundary in eq. (2.68) has not been made explicit (doing so is likely the main part of the work required to actually prove this result).

2.5.4 Alternate duality identities

The divergence equations have been seen to be consequences of Stokes theorem eq. (2.68). One can construct alternate variations that do not have names that we are familiar with.

An example, consider $f = IF$, in the following

$$\iint (\nabla \wedge f) \cdot d\mathbf{A} = \oint f \cdot d\mathbf{r}$$

for $f \in \mathbb{R}^N$. For $f \in \mathbb{R}^2$, the object F is a vector, and gives us the two variable divergence equation, However, the other spaces produce other grade objects and some unfamiliar formulas. For example, F is a bivector in \mathbb{R}^3 , and is a trivector in \mathbb{R}^4 F , and so forth.

$$\begin{aligned}
(\nabla \wedge f) \cdot d\mathbf{A} &= \langle \nabla I F \rangle_2 \cdot d\mathbf{A} \\
&= (-1)^{n-1} \langle I \nabla F \rangle_2 \cdot d\mathbf{A} \\
&\quad n-2 \text{ blade} \\
&= (-1)^{n-1} \left\langle I \left(\overbrace{\nabla \cdot F}^{n-2 \text{ blade}} + \underbrace{\nabla \wedge F}_2 \right) \right\rangle \cdot d\mathbf{A} \\
&= (-1)^{n-1} (I(\nabla \cdot F)) \cdot d\mathbf{A} \\
&= (-1)^{n-1} \langle I(\nabla \cdot F) d\mathbf{A} \rangle \\
&= (-1)^{n-1} \langle (\nabla \cdot F) d\mathbf{A} I \rangle \\
&= (-1)^{n-1} (\nabla \cdot F) \cdot (d\mathbf{A} I) \\
f \cdot d\mathbf{r} &= \langle I F d\mathbf{r} \rangle \\
&= \langle F d\mathbf{r} I \rangle \\
&= (-1)^{n-1} \langle F I d\mathbf{r} \rangle \\
&= (-1)^{n-1} F \cdot (I d\mathbf{r})
\end{aligned} \tag{2.72}$$

$$\begin{aligned}
f \cdot d\mathbf{r} &= \langle I F d\mathbf{r} \rangle \\
&= \langle F d\mathbf{r} I \rangle \\
&= (-1)^{n-1} \langle F I d\mathbf{r} \rangle \\
&= (-1)^{n-1} F \cdot (I d\mathbf{r})
\end{aligned} \tag{2.73}$$

This gives

$$\iint (\nabla \cdot F) \cdot (d\mathbf{A} I) = \oint F \cdot (I d\mathbf{r}) \tag{2.74}$$

As an example, for \mathbb{R}^3 where F is a bivector, $d\mathbf{A} I$ is a two variable parametrized vector “line” element (say ds), and $I d\mathbf{r}$ is a single variable parametrized bivector “surface” element (say $d\mathbf{B}$). This gives us a peculiar (or unfamiliar) looking duality generated identity

$$\iint (\nabla \cdot F) \cdot ds = \int F \cdot d\mathbf{B}$$

2.5.5 Summary remarks

This mostly completes the aim of this examination. There are a couple of identities still to derive in the intro table (equations eq. (2.7), and eq. (2.8)). Everything else has been shown to be a special case of the fundamental exterior derivative integral eq. (2.9), and we have assigned specific meanings and orientations to the spatial volume elements on each side of that equation for the scalar, vector and bivector cases. This may be enough for electromagnetism calculations, unless a trivector and four-space result is also required.

Stokes Law revisited with algebraic enumeration of boundary

3.1 Algebraic description of oriented boundaries

Having used pictorial methods to enumerate the bounding loop and area elements [2](#) in the previous derivation of the vector and bivector forms of Stokes's, makes the application of these formulas harder. Here this will be revisited, with the aim of remedying this, as well as obtaining a proof for the general case, which was not possible because of a lack of exactly this algebraic formulation.

3.1.1 Parallelogram parametrization

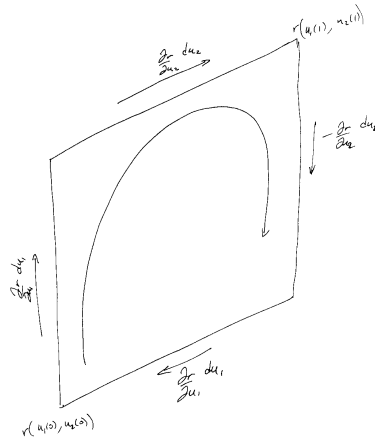


Figure 3.1: Two variable parametrization of \mathbb{R}^n parallelogram

An oriented curve around a parallelogram in \mathbb{R}^n is illustrated in fig. [3.1](#). We want to evaluate the line integral around this path

$$\oint \mathbf{f} \cdot d\mathbf{r} = \int du_1 \mathbf{f} \cdot \frac{\partial \mathbf{r}}{\partial u_1} \Big|_{u_2(0)}^{u_2(1)} - \int du_2 \mathbf{f} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \Big|_{u_1(0)}^{u_1(1)} \quad (3.1)$$

Now, we can put this in a more symmetric form utilizing a reciprocal frame to enumerate the alternation. Write

$$\begin{aligned} \mathbf{r}_{u_i} &= \frac{\partial \mathbf{r}}{\partial u_i} \\ \mathbf{r}^{u_j} \cdot \mathbf{r}_{u_i} &= \delta^j_i \\ I &= \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2} \\ I\mathbf{r}^{u_1} &= I \cdot \mathbf{r}^{u_1} = -\mathbf{r}_{u_2} \\ I\mathbf{r}^{u_2} &= I \cdot \mathbf{r}^{u_2} = \mathbf{r}_{u_1}. \end{aligned} \quad (3.2)$$

We do not care to actually calculate the reciprocal frame vectors. They just work well to describe the alternation in terms of the pseudoscalar for the plane.

Substituting back into eq. (3.1) we have

$$\oint \mathbf{f} \cdot d\mathbf{r} = \int du_1 \mathbf{f} \cdot (I\mathbf{r}^{u_2}) \Big|_{u_2(0)}^{u_2(1)} + \int du_2 \mathbf{f} \cdot (I\mathbf{r}^{u_1}) \Big|_{u_1(0)}^{u_1(1)} \quad (3.3)$$

Or

$$\oint \mathbf{f} \cdot d\mathbf{r} = \sum_i \int \frac{du_1 du_2}{du_i} \mathbf{f} \cdot (I\mathbf{r}^{u_i}) \Big|_{u_i(0)}^{u_i(1)} \quad (3.4)$$

This completes the goal of expressing the line integral in a fashion that does not require drawing any pictures, and gives a hint about how to do the same for general $\wedge^k \mathbb{R}^n$ case.

As before this can be written in terms of its integrals

$$\begin{aligned} \oint \mathbf{f} \cdot d\mathbf{r} &= \sum_{i,j \neq i} \int_{u_j(0)}^{u_j(1)} du_j \int_{u_i(0)}^{u_i(1)} \frac{\partial}{\partial u_i} \mathbf{f} \cdot (I\mathbf{r}^{u_i}) du_i \\ &= \iint du_1 du_2 \sum \frac{\partial}{\partial u_i} \mathbf{f} \cdot (I\mathbf{r}^{u_i}) \end{aligned} \quad (3.5)$$

Evaluating the derivatives to prove the Stokes/Green's result will be deferred for now (may instead proving the general case once formulated).

3.1.2 *Parallelepiped parametrization*

Next, lets evaluate the bivector area dot products, as in fig. 3.2.

$$\begin{aligned} \oiint \mathbf{F} \cdot d\mathbf{A} &= \iint du_2 du_1 F \cdot (\mathbf{r}_{u_2} \wedge \mathbf{r}_{u_1}) \Big|_{u_3(1)}^{u_3(0)} \\ &+ \iint du_3 du_1 F \cdot (\mathbf{r}_{u_3} \wedge \mathbf{r}_{u_1}) \Big|_{u_2(0)}^{u_2(1)} \\ &+ \iint du_3 du_2 F \cdot (\mathbf{r}_{u_3} \wedge -\mathbf{r}_{u_2}) \Big|_{u_1(0)}^{u_1(1)} \end{aligned} \quad (3.6)$$

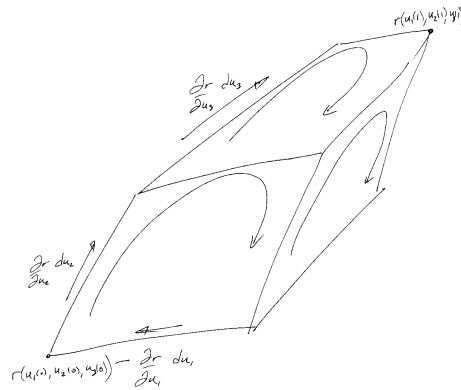


Figure 3.2: Three variable parametrization of \mathbb{R}^n parallelepiped

Again introducing reciprocal vectors to enumerate the alternation, but now write I as a pseudoscalar for the parallelepiped subspace that the area bounds

$$\begin{aligned}
 I &= \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2} \wedge \mathbf{r}_{u_3} \\
 I\mathbf{r}^{u_1} &= \mathbf{r}_{u_2} \wedge \mathbf{r}_{u_3} \\
 I\mathbf{r}^{u_2} &= -\mathbf{r}_{u_1} \wedge \mathbf{r}_{u_3} \\
 I\mathbf{r}^{u_3} &= \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2}
 \end{aligned} \tag{3.7}$$

Substituting we have a form almost identical to the line integral of eq. (3.4).

$$\begin{aligned}
 \oint \mathbf{F} \cdot d\mathbf{A} &= \sum \iint \frac{du_1 du_2 du_3}{du_i} F \cdot (I\mathbf{r}^{u_i}) \Big|_{u_i^{(0)}}^{u_i^{(1)}} \\
 &= \iiint du_1 du_2 du_3 \sum \frac{\partial}{\partial u_i} F \cdot (I\mathbf{r}^{u_i})
 \end{aligned} \tag{3.8}$$

3.1.3 General case

Having found that the line integral and oriented area integrals can be expressed uniformly in the same algebraic form, it is reasonable to define an integral with such structure as a directed hypervolume boundary for any grade blade, and then verify that this yields the expected generalized Stokes result that has been proven for only the vector and area cases.

Writing

$$F \in \bigwedge^{k-1} \mathbb{R}^n$$

$$d^k \mathbf{x} = \frac{\partial \mathbf{r}}{\partial u_1} \wedge \frac{\partial \mathbf{r}}{\partial u_2} \wedge \cdots \wedge \frac{\partial \mathbf{r}}{\partial u_k} du_1 du_2 \cdots du_k = I du_1 du_2 \cdots du_k \quad (3.9)$$

We wish to prove the general Stokes equation for a hyper-parallelepiped volume

$$\int_V (\nabla \wedge F) \cdot d^k \mathbf{x} = \int_{\partial V} F \cdot d^{k-1} \mathbf{x} \quad (3.10)$$

With the presumption that this will algebraically be identical to the line integral and area integral cases for vectors and bivectors respectively we want to evaluate

$$\int_{\partial V} F \cdot d^{k-1} \mathbf{x} = \sum \int \frac{du_1 du_2 \cdots du_k}{du_i} F \cdot (\mathbf{I} \mathbf{r}^{u_i}) \Big|_{u_i(0)}^{u_i(1)}$$

$$= \int_V du_1 du_2 \cdots du_k \sum \frac{\partial}{\partial u_i} F \cdot (\mathbf{I} \mathbf{r}^{u_i}) \quad (3.11)$$

$$\int_{\partial V} F \cdot d^{k-1} \mathbf{x} = \int_V du_1 du_2 \cdots du_k \sum \frac{\partial F}{\partial u_i} \cdot (\mathbf{I} \mathbf{r}^{u_i}) + F \cdot \left(\frac{\partial}{\partial u_i} \mathbf{I} \mathbf{r}^{u_i} \right). \quad (3.12)$$

The last term here sums to zero. The messy long proof of this can be found at the end. Assuming that proven this leaves us with the following identity

$$\int_{\partial V} F \cdot d^{k-1} \mathbf{x} = \int_V du_1 du_2 \cdots du_k \sum \frac{\partial F}{\partial u_i} \cdot (\mathbf{I} \mathbf{r}^{u_i}) \quad (3.13)$$

We wish to show that this equals

$$\int_V du_1 du_2 \cdots du_k (\nabla \wedge F) \cdot I,$$

after which point we have both formulated algebraically the boundary integral, and proven the general $k - 1$ blade Stokes theorem of eq. (3.10).

3.1.4 *Is a coordinate free proof possible?*

Note that

$$\begin{aligned} \frac{\partial F}{\partial u_i} \cdot (\mathbf{I} \mathbf{r}^{u_i}) &= \left\langle \frac{\partial F}{\partial u_i} \mathbf{I} \mathbf{r}^{u_i} \right\rangle \\ &= \left\langle \mathbf{r}^{u_i} \frac{\partial F}{\partial u_i} I \right\rangle \\ &= \left(\mathbf{r}^{u_i} \wedge \frac{\partial F}{\partial u_i} \right) \cdot I \end{aligned} \quad (3.14)$$

Can the reduction of this wedge product to curl form be done without coordinates? It would also be fairly easy to go in circles here since the reciprocal frame vectors can be calculated in terms of the pseudoscalar I .

3.1.5 Notation for coordinate expansion

I did not have any luck finding a coordinate free way as outlined above to prove the general result. The dumb brute force way is still possible though, expanding both sides and comparing.

The following will be used in the sections below

$$\begin{aligned}\mathbf{r} &= \gamma_j x^j \\ \mathbf{r}_{u_i} &= \gamma_j \frac{\partial x^j}{\partial u_i}\end{aligned}\tag{3.15}$$

$$\begin{aligned}I\mathbf{r}^{u_i} &= (-1)^{k-i} \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2} \wedge \cdots \widehat{\mathbf{r}_{u_i}} \cdots \wedge \mathbf{r}_{u_k} \\ I\mathbf{r}^{u_i} &= (-1)^{k-i} \gamma_{j_1} \wedge \cdots \wedge \gamma_{j_{k-1}} \frac{\partial x^{j_1}}{\partial u_1} \cdots \widehat{\frac{\partial}{\partial u_i}} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k}\end{aligned}\tag{3.16}$$

Here the overhat is used to indicate omission.

3.1.6 Expanding the curl dot by coordinates

One half of the comparison will be based on the expansion of $(\nabla \wedge F) \cdot I$. We calculate

$$\begin{aligned}F &= \frac{1}{(k-1)!} F_{j_1 j_2 \cdots j_{k-1}} \gamma^{j_1} \wedge \gamma^{j_2} \cdots \wedge \gamma^{j_{k-1}} \\ I &= \gamma_{m_1} \wedge \gamma_{m_2} \cdots \wedge \gamma_{m_k} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \cdots \frac{\partial x^{m_k}}{\partial u_k} \\ \nabla &= \gamma^{j_k} \frac{\partial}{\partial x^{j_k}} \\ \nabla \wedge F &= \frac{1}{(k-1)!} \frac{\partial F_{j_1 j_2 \cdots j_{k-1}}}{\partial x^{j_k}} \gamma^{j_k} \wedge \gamma^{j_1} \wedge \gamma^{j_2} \cdots \wedge \gamma^{j_{k-1}} \\ &= \frac{(-1)^{k-1}}{(k-1)!} \frac{\partial F_{j_1 j_2 \cdots j_{k-1}}}{\partial x^{j_k}} \gamma^{j_1} \wedge \gamma^{j_2} \cdots \wedge \gamma^{j_{k-1}} \wedge \gamma^{j_k}.\end{aligned}\tag{3.17}$$

Now, put this all together

$$\begin{aligned}(\nabla \wedge F) \cdot I &= \frac{(-1)^{k-1}}{(k-1)!} (\gamma^{j_1} \wedge \gamma^{j_2} \cdots \wedge \gamma^{j_k}) \cdot (\gamma_{m_1} \wedge \gamma_{m_2} \cdots \wedge \gamma_{m_k}) \\ &= \frac{\partial F_{j_1 j_2 \cdots j_{k-1}}}{\partial x^{j_k}} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \cdots \frac{\partial x^{m_k}}{\partial u_k} \\ &= \frac{(-1)^{k-1}}{(k-1)!} \delta^{j_k}_{m_1} \delta^{j_{k-1}}_{m_2} \cdots \delta^{j_1}_{m_k} \epsilon^{m_1 m_2 \cdots m_k} \frac{\partial F_{j_1 j_2 \cdots j_{k-1}}}{\partial x^{j_k}} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \cdots \frac{\partial x^{m_k}}{\partial u_k} \\ &= \frac{(-1)^{k-1}}{(k-1)!} \epsilon^{m_1 m_2 \cdots m_k} \frac{\partial F_{m_k m_{k-1} \cdots m_2}}{\partial x^{m_1}} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \cdots \frac{\partial x^{m_k}}{\partial u_k}\end{aligned}\tag{3.18}$$

Now, to reverse a k vector, or its corresponding antisymmetric tensor as above we have to perform the following number of swaps

$$k-1+k-2+\dots+1=k(k-1)/2$$

We can use this to tidy the indices above

$$k-1+(k-1)(k-2)/2=k(k-1)/2,$$

and thus write

$$(\nabla \wedge F) \cdot I = \frac{(-1)^{k(k-1)/2}}{(k-1)!} \epsilon^{m_1 m_2 \dots m_k} \frac{\partial F_{m_2 \dots m_k}}{\partial x^{m_1}} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \dots \frac{\partial x^{m_k}}{\partial u_k} \quad (3.19)$$

3.1.7 Expanding the boundary integral by coordinates

The remainder of the proof is to verify that the expression eq. (3.19) matches the differential form in eq. (3.13).

To do so we have to expand

$$\begin{aligned} & \frac{\partial F}{\partial u_i} \cdot (I \mathbf{r}^{u_i}) \\ \frac{\partial F}{\partial u_i} &= \frac{\partial x^{m_1}}{\partial u_i} \frac{\partial}{\partial x^{m_1}} \frac{1}{(k-1)!} F_{m_2 m_3 \dots m_k} \gamma^{m_2} \wedge \gamma^{m_3} \dots \wedge \gamma^{m_k} \end{aligned} \quad (3.20)$$

Dotting this with eq. (3.16) we have

$$\begin{aligned} \frac{\partial F}{\partial u_i} \cdot (I \mathbf{r}^{u_i}) &= \frac{(-1)^{k-i}}{(k-1)!} \frac{\partial F_{m_2 m_3 \dots m_k}}{\partial x^{m_1}} (\gamma^{m_2} \wedge \gamma^{m_3} \dots \wedge \gamma^{m_k}) \cdot (\gamma_{j_1} \wedge \dots \wedge \gamma_{j_{k-1}}) \\ &= \frac{\partial x^{j_1}}{\partial u_1} \dots \widehat{\frac{\partial}{\partial u_i}} \dots \frac{\partial x^{j_{k-1}}}{\partial u_{k-1}} \frac{\partial x^{m_1}}{\partial u_i} \\ &= \frac{(-1)^{k-i}}{(k-1)!} \frac{\partial F_{m_2 m_3 \dots m_k}}{\partial x^{m_1}} \delta^{m_k}_{j_1} \delta^{m_{k-1}}_{j_2} \dots \delta^{m_2}_{j_{k-1}} \epsilon^{j_1 j_2 \dots j_{k-1}} \\ &= \frac{\partial x^{j_1}}{\partial u_1} \dots \widehat{\frac{\partial}{\partial u_i}} \dots \frac{\partial x^{j_{k-1}}}{\partial u_{k-1}} \frac{\partial x^{m_1}}{\partial u_i} \\ &= \frac{(-1)^{k-i}}{(k-1)!} \frac{\partial F_{m_2 m_3 \dots m_k}}{\partial x^{m_1}} \epsilon^{m_k m_{k-1} \dots m_2} \frac{\partial x^{m_k}}{\partial u_1} \dots \widehat{\frac{\partial}{\partial u_i}} \dots \frac{\partial x^{m_2}}{\partial u_k} \frac{\partial x^{m_1}}{\partial u_i} \\ &= \frac{-(-1)^{i+k(k-1)/2}}{(k-1)!} \frac{\partial F_{m_2 m_3 \dots m_k}}{\partial x^{m_1}} \epsilon^{m_2 \dots m_{k-1}} \frac{\partial x^{m_k}}{\partial u_1} \dots \widehat{\frac{\partial}{\partial u_i}} \dots \frac{\partial x^{m_2}}{\partial u_k} \frac{\partial x^{m_1}}{\partial u_i} \end{aligned} \quad (3.21)$$

After nicely arranging the m_i indices to match eq. (3.19), the partials do not match. Perhaps about a change of variables:

$$\begin{aligned} m_1 &= n_i \\ m_k &= n_1 \\ m_{k-1} &= n_2 \\ &\dots \dots \\ m_2 &= n_k \end{aligned} \quad (3.22)$$

(with appropriate adjustments for $i=1$)

$$\begin{aligned} \frac{\partial F}{\partial u_i} \cdot (I\mathbf{r}^{u_i}) &= \frac{-(-1)^{i+k(k-1)/2}}{(k-1)!} \frac{\partial F_{n_k n_{k-1} \dots \widehat{n_i} \dots n_1}}{\partial x^{n_i}} \epsilon^{n_k \dots \widehat{n_i} \dots n_1} \frac{\partial x^{n_1}}{\partial u_1} \frac{\partial x^{n_2}}{\partial u_2} \dots \frac{\partial x^{n_k}}{\partial u_k} \\ &= \frac{-(-1)^{i+k(k-1)/2}}{(k-1)!} \frac{\partial F_{n_1 n_2 \dots \widehat{n_i} \dots n_k}}{\partial x^{n_i}} \epsilon^{n_1 \dots \widehat{n_i} \dots n_k} \frac{\partial x^{n_1}}{\partial u_1} \frac{\partial x^{n_2}}{\partial u_2} \dots \frac{\partial x^{n_k}}{\partial u_k} \end{aligned} \quad (3.23)$$

Here we have the product of two completely antisymmetric tensors, both with the same set of indices, so any alternation of those indices has no effect. The only sign changes come from the $-(-1)^i$ coefficient.

To verify consistency with eq. (3.19) it remains to prove that within the sum the following two are identical

$$\epsilon^{m_1 m_2 \dots m_k} \frac{\partial F_{m_2 \dots m_k}}{\partial x^{m_1}} \quad (3.24)$$

$$(-1)^{i+1} \frac{\partial F_{n_1 n_2 \dots \widehat{n_i} \dots n_k}}{\partial x^{n_i}} \epsilon^{n_1 \dots \widehat{n_i} \dots n_k}. \quad (3.25)$$

Examination and a bit of thought shows this to be the case. FIXME: this statement is intuition based, and I am having trouble describing exactly why I say so. Revisit this later (for now I had rather spend the time working with the result than to complete the last details of the proof).

3.2 Summary

Summarizing, a proof has been given for the general multivector Stokes equation, that provides equivalent volume and boundary integral expressions

$$\int_V (\nabla \wedge F) \cdot d^k \mathbf{x} = \int_{\partial V} F \cdot d^{k-1} \mathbf{x}. \quad (3.26)$$

The proof of this result was restricted to a hyper-parallelepiped volume and its corresponding boundary. Additional arguments are required to extend this to arbitrary shapes. That argument follows the loop integral case where cancellation of oppositely oriented surfaces in adjacent volumes can be used to build up an arbitrary shape in terms of small parallelepiped volumes.

In addition to the proof of this result, a specific algebraic (non-pictorial) meaning has been given to the boundary differential form $d^{k-1} \mathbf{x}$. We have used the following notation

$$\mathbf{r}_{u_i} = \frac{\partial \mathbf{r}}{\partial u_i} \quad (3.27)$$

$$\mathbf{r}^{u_i} \cdot \mathbf{r}_{u_j} = \delta^i_j \quad (3.28)$$

$$\begin{aligned} I &= \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2} \wedge \dots \wedge \mathbf{r}_{u_k} \\ &= \gamma_{m_1} \wedge \gamma_{m_2} \dots \wedge \gamma_{m_k} \frac{\partial x^{m_1}}{\partial u_1} \frac{\partial x^{m_2}}{\partial u_2} \dots \frac{\partial x^{m_k}}{\partial u_k} \end{aligned} \quad (3.29)$$

$$\begin{aligned} I\mathbf{r}^{u_i} &= I \cdot \mathbf{r}^{u_i} \\ &= (-1)^{k-i} \mathbf{r}_{u_1} \wedge \mathbf{r}_{u_2} \wedge \dots \wedge \widehat{\mathbf{r}_{u_i}} \dots \wedge \mathbf{r}_{u_k} \\ &= (-1)^{k-i} \gamma_{j_1} \wedge \dots \wedge \gamma_{j_{k-1}} \frac{\partial x^{j_1}}{\partial u_1} \dots \frac{\partial}{\partial u_i} \dots \frac{\partial x^{j_{k-1}}}{\partial u_k} \end{aligned} \quad (3.30)$$

Here \mathbf{r} , as parametrized by u_i spans the hyper-parallelepiped, and $\mathbf{r}(u_1, \dots, u_i(1), \dots)$, and $\mathbf{r}(u_1, \dots, u_i(0), \dots)$ represent boundaries of the surface with respect to parameter u_i . Putting things together we have the following algebraic description of the boundary

$$\int_{\partial V} F \cdot d^{k-1} \mathbf{x} = \sum \int du_1 \cdots \widehat{du}_i \cdots du_k F \cdot (\mathbf{I} \mathbf{r}^{u_i}) \Big|_{u_i(0)}^{u_i(1)} \quad (3.31)$$

Observe that we have a Jacobian like relationship above due to the alternation provided by the wedge product. For this reason it would make sense to introduce vector differentials

$$d\mathbf{x}_i = \frac{\partial \mathbf{r}}{\partial u_i} du_i, \quad (3.32)$$

in order to suppress the explicit parametrization.

$$\begin{aligned} \int_{\partial V} F \cdot d^{k-1} \mathbf{x} &= \int \sum (-1)^{k-i} F \cdot (d\mathbf{x}_1 \wedge \cdots \widehat{d\mathbf{x}}_i \cdots \wedge d\mathbf{x}_k) \Big|_{u_i(0)}^{u_i(1)} \\ &= \int \sum (-1)^{k-i} F \cdot (d\mathbf{x}_1 \wedge \cdots \widehat{d\mathbf{x}}_i \cdots \wedge d\mathbf{x}_k) \Big|_{\partial \mathbf{x}_i} \end{aligned} \quad (3.33)$$

In the LHS of eq. (3.26) we also have a specific meaning for the k-vector volume element

$$\begin{aligned} d^k \mathbf{x} &= I du_1 du_2 \cdots du_k \\ &= d\mathbf{x}_1 \wedge d\mathbf{x}_2 \cdots \wedge d\mathbf{x}_k \end{aligned} \quad (3.34)$$

Also notable for the volume integral is its tensor formulation, where we have the volume Jacobian determinant explicitly

$$\int_V (\nabla \wedge F) \cdot d^k \mathbf{x} = \int_V \frac{(-1)^{k(k-1)/2}}{(k-1)!} \frac{\partial F_{m_2 \cdots m_k}}{\partial x^{m_1}} \frac{\partial (x^{m_1}, \dots, x^{m_k})}{\partial (u_1, \dots, u_k)} du_1 \cdots du_k \quad (3.35)$$

The other interesting thing worth noting is the reciprocal expression for curl projected onto the integration subspace

$$(\nabla \wedge F) \cdot I = \left(\sum_i \mathbf{r}^{u_i} \wedge \frac{\partial F}{\partial u_i} \right) \cdot I \quad (3.36)$$

I was not able to use this, but having mostly completed the proof, this is proved as a side effect. Here mostly means that the unsatisfactory treatments (really handwaving) marked with FIXMEs should be revisited to consider this multivector form of Stokes theorem fully proved here.

3.3 Messy proof of zero sum

Sum of eq. (3.12) requires followup.

Here is the deferred proof that the sum of the differentials of the area elements are zero

$$\int du_1 du_2 \cdots du_k \sum F \cdot \left(\frac{\partial}{\partial u_i} \mathbf{I} \mathbf{r}^{u_i} \right). \quad (3.37)$$

Although not elegant, the partials here can be expanded by coordinates as done in the previous line and area proofs.

We want to prove that

$$\sum (-1)^{k-i} \gamma_{j_1} \wedge \cdots \gamma_{j_{k-1}} \frac{\partial}{\partial u_i} \frac{\partial x^{j_1}}{\partial u_1} \cdots \widehat{\frac{\partial}{\partial u_i}} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k} = 0 \quad (3.38)$$

as was done previously in the vector and bivector cases. Pick as an example the $i = 3$ case, and assume that $k > 2$ since the two simpler cases have been proven explicitly. For that i , we have the following terms

$$\begin{aligned} \sum (-1)^{k-3} \gamma_{j_1} \wedge \cdots \gamma_{j_{k-1}} & \left(\frac{\partial^2 x^{j_1}}{\partial u_3 \partial u_1} \frac{\partial x^{j_2}}{\partial u_2} \frac{\partial x^{j_3}}{\partial u_4} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k} \right. \\ & + \frac{\partial^2 x^{j_2}}{\partial u_3 \partial u_2} \frac{\partial x^{j_1}}{\partial u_1} \frac{\partial x^{j_3}}{\partial u_4} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k} \\ & + \frac{\partial^2 x^{j_3}}{\partial u_3 \partial u_4} \frac{\partial x^{j_1}}{\partial u_1} \frac{\partial x^{j_2}}{\partial u_2} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k} \\ & + \cdots \\ & \left. + \frac{\partial^2 x^{j_k}}{\partial u_3 \partial u_k} \frac{\partial x^{j_1}}{\partial u_1} \frac{\partial x^{j_2}}{\partial u_2} \cdots \frac{\partial x^{j_{k-2}}}{\partial u_{k-1}} \right) \end{aligned} \quad (3.39)$$

Picking any mixed partial term we expect cancellation with the opposing mixed partial. Two representative values of i should be sufficient to see that the sum is zero. First pick $i = 1$, so that $(-1)^{k-3} = (-1)^{k-1}$, and look at the matching partial for the $\frac{\partial^2}{\partial u_1 \partial u_3}$ term above

$$\begin{aligned} \sum (-1)^{k-1} \gamma_{j_1} \wedge \cdots \gamma_{j_{k-1}} & \left(\frac{\partial^2 x^{j_1}}{\partial u_1 \partial u_2} \frac{\partial x^{j_2}}{\partial u_3} \frac{\partial x^{j_3}}{\partial u_4} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k} \right. \\ & + \frac{\partial^2 x^{j_2}}{\partial u_1 \partial u_3} \frac{\partial x^{j_1}}{\partial u_2} \frac{\partial x^{j_3}}{\partial u_4} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k} \\ & + \cdots \\ & \left. + \frac{\partial^2 x^{j_k}}{\partial u_1 \partial u_k} \frac{\partial x^{j_1}}{\partial u_2} \frac{\partial x^{j_2}}{\partial u_3} \cdots \frac{\partial x^{j_{k-2}}}{\partial u_{k-1}} \right). \end{aligned} \quad (3.40)$$

Swapping dummy indices j_1 and j_2 here one can see that the $\frac{\partial^2}{\partial u_1 \partial u_3}$ and $\frac{\partial^2}{\partial u_3 \partial u_1}$ terms cancel.

Now pick $i = 2$, so that $(-1)^{k-1} = -(-1)^{k-2}$, and look at the matching partial for the $\frac{\partial^2}{\partial u_1 \partial u_2}$ term above.

$$\begin{aligned} \sum (-1)^{k-2} \gamma_{j_1} \wedge \cdots \gamma_{j_{k-1}} & \left(\frac{\partial^2 x^{j_1}}{\partial u_2 \partial u_1} \frac{\partial x^{j_2}}{\partial u_3} \frac{\partial x^{j_3}}{\partial u_4} \cdots \frac{\partial x^{j_{k-1}}}{\partial u_k} \right. \\ & + \cdots \\ & \left. + \frac{\partial^2 x^{j_k}}{\partial u_2 \partial u_k} \frac{\partial x^{j_1}}{\partial u_1} \frac{\partial x^{j_2}}{\partial u_3} \cdots \frac{\partial x^{j_{k-2}}}{\partial u_{k-1}} \right). \end{aligned} \quad (3.41)$$

No swap of indices is required and we see again that the mixed partials cancel. Now this is perhaps a slightly lazy proof, but working with indices in the abstract without assigning specific numbers gets confusing. It is clear to me that the end result will be a zero sum for this term.

FIXME: A cleanup of this proof should be possible to eliminate the special case comparisons above. The tough part is simply writing all the terms in a manipulatable fashion. Then proceed to split the sum into terms that differ by even and odd separation of indices. Summing over indices greater and indices lesser, then swapping indices as appropriate should complete the proof. Alternatively, perhaps I will figure out a clever way later to demonstrate this more directly without resorting to this messy coordinate expansion.

Stokes theorem applied to vector and bivector fields

4.1 Vector Stokes Theorem

I found my self forgetting stokes theorem once again. Redo this for the simplest case of a parallelogram area element.

What I recall is that we have on one side the curl dotted into the plane of the surface area element

$$\int (\nabla \wedge A) \cdot d^2x \quad (4.1)$$

and on the other side a loop integral

$$\oint A \cdot dx \quad (4.2)$$

Comparing the two we should end up with the same form and thus determine the form of the grade two Stokes equation (i.e. for curl of a vector).

4.1.1 Bivector product part

$$\begin{aligned} (\nabla \wedge A) \cdot d^2x &= (\nabla \wedge A) \cdot \left(\frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \right) d\alpha d\beta \\ &= \partial_\mu A_\nu \frac{\partial x^\sigma}{\partial \alpha} \frac{\partial x^\epsilon}{\partial \beta} (\gamma^\mu \wedge \gamma^\nu) \cdot (\gamma_\sigma \wedge \gamma_\epsilon) d\alpha d\beta \\ &= \partial_\mu A_\nu \frac{\partial x^\sigma}{\partial \alpha} \frac{\partial x^\epsilon}{\partial \beta} (\delta^\mu_\epsilon \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\epsilon) d\alpha d\beta \\ &= \partial_\mu A_\nu \left(\frac{\partial x^\nu}{\partial \alpha} \frac{\partial x^\mu}{\partial \beta} - \frac{\partial x^\mu}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \right) d\alpha d\beta \end{aligned} \quad (4.3)$$

So we have

$$(\nabla \wedge A) \cdot d^2x = -\partial_\mu A_\nu \frac{\partial(x^\mu, x^\nu)}{\partial(\alpha, \beta)} d\alpha d\beta \quad (4.4)$$

4.1.2 Loop integral part

Integrating around a parallelogram spacetime area element with sides $d\alpha\partial x/\partial\alpha$ and $d\beta\partial x/\partial\beta$, as depicted in fig. 4.1, we have

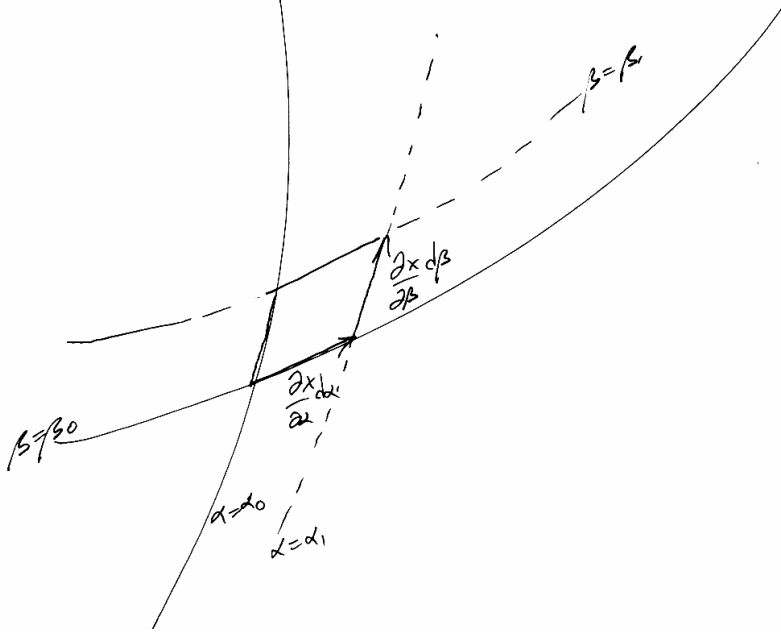


Figure 4.1: Surface area element

$$\begin{aligned}
 \oint A \cdot dx &= \int A|_{\beta=\beta_0} \cdot \frac{\partial x}{\partial \alpha} d\alpha + A|_{\alpha=\alpha_1} \cdot \frac{\partial x}{\partial \beta} d\beta + A|_{\beta=\beta_1} \cdot \left(-\frac{\partial x}{\partial \alpha} d\alpha\right) + A|_{\alpha=\alpha_0} \cdot \left(-\frac{\partial x}{\partial \beta} d\beta\right) \\
 &= \int (A|_{\alpha=\alpha_1} - A|_{\alpha=\alpha_0}) \cdot \frac{\partial x}{\partial \beta} d\beta - (A|_{\beta=\beta_1} - A|_{\beta=\beta_0}) \cdot \frac{\partial x}{\partial \alpha} d\alpha \\
 &= \int \frac{\partial A}{\partial \alpha} \cdot \frac{\partial x}{\partial \beta} d\alpha d\beta - \frac{\partial A}{\partial \beta} \cdot \frac{\partial x}{\partial \alpha} d\beta d\alpha
 \end{aligned} \tag{4.5}$$

Expanding the derivatives in terms of coordinates we have

$$\begin{aligned}
 \frac{\partial A}{\partial \sigma} &= \frac{\partial A_\mu}{\partial \sigma} \gamma^\mu \\
 &= \frac{\partial A_\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial \sigma} \gamma^\mu \\
 &= \partial_\nu A_\mu \frac{\partial x^\nu}{\partial \sigma} \gamma^\mu
 \end{aligned} \tag{4.6}$$

and

$$\frac{\partial x}{\partial \sigma} = \frac{\partial x^\nu}{\partial \sigma} \gamma_\nu \tag{4.7}$$

Assembling we have

$$\oint A \cdot dx = \int \partial_\nu A_\mu \left(\frac{\partial x^\nu}{\partial \alpha} \frac{\partial x^\mu}{\partial \beta} - \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\mu}{\partial \alpha} \right) d\alpha d\beta \quad (4.8)$$

In terms of the Jacobian used in eq. (4.4) we have

$$\oint A \cdot dx = \int \partial_\mu A_\nu \frac{\partial(x^\mu, x^\nu)}{\partial(\alpha, \beta)} d\alpha d\beta \quad (4.9)$$

Comparing the two we have only a sign difference so the conclusion is that Stokes for a vector field (considering only a flat parallelogram area element) is

$$\int (\nabla \wedge A) \cdot d^2x = \oint A \cdot dx \quad (4.10)$$

Observe that there is an implied orientation of the area element on the LHS, required to match up with the orientation of the RHS integral.

4.2 Bivector Stokes Theorem

A parallelepiped volume element is depicted in fig. 4.2. Three parameters α, β, σ generate a set of differential vector displacements spanning the three dimensional subspace

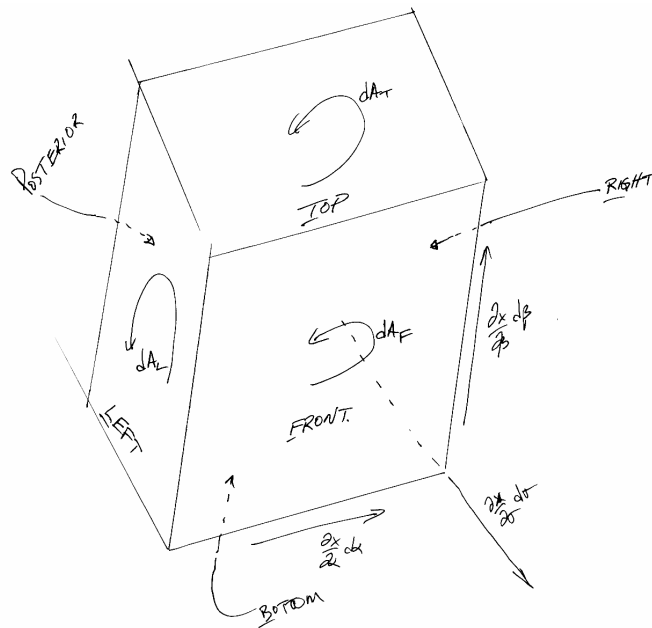


Figure 4.2: Differential volume element

Writing the displacements

$$\begin{aligned}
dx_\alpha &= \frac{\partial x}{\partial \alpha} d\alpha \\
dx_\beta &= \frac{\partial x}{\partial \beta} d\beta \\
dx_\sigma &= \frac{\partial x}{\partial \sigma} d\sigma
\end{aligned} \tag{4.11}$$

We have for the front, right and top face area elements

$$\begin{aligned}
dA_F &= dx_\alpha \wedge dx_\beta \\
dA_R &= dx_\beta \wedge dx_\sigma \\
dA_T &= dx_\sigma \wedge dx_\alpha
\end{aligned} \tag{4.12}$$

These are the surfaces of constant parametrization, respectively, $\sigma = \sigma_1$, $\alpha = \alpha_1$, and $\beta = \beta_1$. For a bivector, the flux through the surface is therefore

$$\begin{aligned}
\int B \cdot dA &= (B_{\sigma_1} \cdot dA_F - B_{\sigma_0} \cdot dA_P) + (B_{\alpha_1} \cdot dA_R - B_{\alpha_0} \cdot dA_L) + (B_{\beta_1} \cdot dA_T - B_{\beta_0} \cdot dA_B) \\
&= d\sigma \frac{\partial B}{\partial \sigma} \cdot (dx_\alpha \wedge dx_\beta) + d\alpha \frac{\partial B}{\partial \alpha} \cdot (dx_\beta \wedge dx_\sigma) + d\beta \frac{\partial B}{\partial \beta} \cdot (dx_\sigma \wedge dx_\alpha)
\end{aligned} \tag{4.13}$$

Written out in full this is a bit of a mess

$$\int B \cdot dA = d\alpha d\beta d\sigma \partial_\mu B \cdot \left(\left(-\frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \alpha} + \frac{\partial x^\mu}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} + \frac{\partial x^\mu}{\partial \beta} \frac{\partial x^\nu}{\partial \sigma} \frac{\partial x^\epsilon}{\partial \alpha} \right) (\gamma_\nu \wedge \gamma_\epsilon) \right) \tag{4.14}$$

It should equal, at least up to a sign, $\int (\nabla \wedge B) \cdot d^3x$. Expanding the latter is probably easier than regrouping the mess, and doing so we have

$$\begin{aligned}
(\nabla \wedge B) \cdot d^3x &= d\alpha d\beta d\sigma (\gamma^\mu \wedge \partial_\mu B) \cdot \left(\frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \\
&= d\alpha d\beta d\sigma \frac{1}{2} (\gamma^\mu \partial_\mu B + \partial_\mu B \gamma^\mu) \cdot \left(\frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \\
&= d\alpha d\beta d\sigma \frac{1}{2} \left\langle (\gamma^\mu \partial_\mu B + \partial_\mu B \gamma^\mu) \left(\frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \right\rangle \\
&= d\alpha d\beta d\sigma \frac{1}{2} \partial_\mu B \cdot \left\langle \left(\frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \gamma^\mu + \gamma^\mu \left(\frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \right\rangle_2 \\
&= d\alpha d\beta d\sigma \partial_\mu B \cdot \left(\left(\frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \cdot \gamma^\mu \right)
\end{aligned} \tag{4.15}$$

Expanding just that trivector-vector dot product

$$\begin{aligned}
\left(\frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \cdot \gamma^\mu &= \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} (\gamma_\lambda \wedge \gamma_\nu \wedge \gamma_\epsilon) \cdot \gamma^\mu \\
&= \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} (\gamma_\lambda \wedge \gamma_\nu \delta_\epsilon^\mu - \gamma_\lambda \wedge \gamma_\epsilon \delta_\nu^\mu + \gamma_\nu \wedge \gamma_\epsilon \delta_\lambda^\mu)
\end{aligned} \tag{4.16}$$

So we have

$$\begin{aligned}
(\nabla \wedge B) \cdot d^3x &= d\alpha d\beta d\sigma \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} \partial_\mu B \cdot (\gamma_\lambda \wedge \gamma_\nu \delta_\epsilon^\mu - \gamma_\lambda \wedge \gamma_\epsilon \delta_\nu^\mu + \gamma_\nu \wedge \gamma_\epsilon \delta_\lambda^\mu) \\
&= d\alpha d\beta d\sigma \partial_\mu B \cdot \left(\frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\mu}{\partial \sigma} \gamma_\lambda \wedge \gamma_\nu + \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\mu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} \gamma_\epsilon \wedge \gamma_\lambda + \frac{\partial x^\mu}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} \gamma_\nu \wedge \gamma_\epsilon \right) \quad (4.17) \\
&= d\alpha d\beta d\sigma \partial_\mu B \cdot \left(\left(\frac{\partial x^\nu}{\partial \alpha} \frac{\partial x^\epsilon}{\partial \beta} \frac{\partial x^\mu}{\partial \sigma} + \frac{\partial x^\epsilon}{\partial \alpha} \frac{\partial x^\mu}{\partial \beta} \frac{\partial x^\nu}{\partial \sigma} + \frac{\partial x^\mu}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} \right) \gamma_\nu \wedge \gamma_\epsilon \right)
\end{aligned}$$

Noting that an ϵ, ν interchange in the first term inverts the sign, we have an exact match with eq. (4.14), thus fixing the sign for the bivector form of Stokes theorem for the orientation picked in this diagram

$$\int (\nabla \wedge B) \cdot d^3x = \int B \cdot d^2x \quad (4.18)$$

Like the vector case, there is a requirement to be very specific about the meaning given to the oriented surfaces, and the corresponding oriented volume element (which could be a volume subspace of a greater than three dimensional space).

Stokes theorem. Revisited again

5.1 Motivation

Relying on pictorial means and a brute force ugly comparison of left and right hand sides, a verification of Stokes theorem for the vector and bivector cases was performed (4). This was more of a confirmation than a derivation, and the technique fails the transition to the trivector case. The trivector case is of particular interest in electromagnetism since that and a duality transformation provides a four-vector divergence theorem.

The fact that the pictorial means of defining the boundary surface does not work well in four vector space is not the only unsatisfactory aspect of the previous treatment. The fact that a coordinate expansion of the hypervolume element and hypersurface element was performed in the LHS and RHS comparisons was required is particularly ugly. It is a lot of work and essentially has to be undone on the opposing side of the equation. Comparing to previous attempts to come to terms with Stokes theorem in (2) and (3) this more recent attempt at least avoids the requirement for a tensor expansion of the vector or bivector. It should be possible to build on this and minimize the amount of coordinate expansion required and go directly from the volume integral to the expression of the boundary surface.

5.2 Do it

5.2.1 Notation and Setup

The desire is to relate the curl hypervolume integral to a hypersurface integral on the boundary

$$\int (\nabla \wedge F) \cdot d^k x = \int F \cdot d^{k-1} x \quad (5.1)$$

In order to put meaning to this statement the volume and surface elements need to be properly defined. In order that this be a scalar equation, the object F in the integral is required to be of grade $k - 1$, and $k \leq n$ where n is the dimension of the vector space that generates the object F .

5.2.2 Reciprocal frames

As evident in equation eq. (5.1) a metric is required to define the dot product. If an affine non-metric formulation of Stokes theorem is possible it will not be attempted here. A reciprocal basis pair will be utilized, defined by

$$\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu \quad (5.2)$$

Both of the sets $\{\gamma_\mu\}$ and $\{\gamma^\mu\}$ are taken to span the space, but are not required to be orthogonal. The notation is consistent with the Dirac reciprocal basis, and there will not be anything in this treatment that prohibits the Minkowski metric signature required for such a relativistic space.

Vector decomposition in terms of coordinates follows by taking dot products. We write

$$x = x^\mu \gamma_\mu = x_\nu \gamma^\nu \quad (5.3)$$

5.2.3 Gradient

When working with a non-orthonormal basis, use of the reciprocal frame can be utilized to express the gradient.

$$\nabla \equiv \gamma^\mu \partial_\mu \equiv \sum_\mu \gamma^\mu \frac{\partial}{\partial x^\mu} \quad (5.4)$$

This contains what may perhaps seem like an odd seeming mix of upper and lower indices in this definition. This is how the gradient is defined in [1]. Although it is possible to accept this definition and work with it, this form can be justified by require of the gradient consistency with the the definition of directional derivative. A definition of the directional derivative that works for single and multivector functions, in \mathbb{R}^3 and other more general spaces is

$$a \cdot \nabla F \equiv \lim_{\lambda \rightarrow 0} \frac{F(x + a\lambda) - F(x)}{\lambda} = \left. \frac{\partial F(x + a\lambda)}{\partial \lambda} \right|_{\lambda=0} \quad (5.5)$$

Taylor expanding about $\lambda = 0$ in terms of coordinates we have

$$\begin{aligned} \left. \frac{\partial F(x + a\lambda)}{\partial \lambda} \right|_{\lambda=0} &= a^\mu \frac{\partial F}{\partial x^\mu} \\ &= (a^\nu \gamma_\nu) \cdot (\gamma^\mu \partial_\mu) F \\ &= a \cdot \nabla F \quad \square \end{aligned} \quad (5.6)$$

The lower index representation of the vector coordinates could also have been used, so using the directional derivative to imply a definition of the gradient, we have an additional alternate representation of the gradient

$$\nabla \equiv \gamma_\mu \partial^\mu \equiv \sum_\mu \gamma_\mu \frac{\partial}{\partial x_\mu} \quad (5.7)$$

5.2.4 Volume element

We define the hypervolume in terms of parametrized vector displacements $x = x(a_1, a_2, \dots, a_k)$. For the vector x we can form a pseudoscalar for the subspace spanned by this parametrization by wedging the displacements in each of the directions defined by variation of the parameters. For $m \in [1, k]$ let

$$dx_i = \frac{\partial x}{\partial a_i} da_i = \gamma_\mu \frac{\partial x^\mu}{\partial a_i} da_i, \quad (5.8)$$

so the hypervolume element for the subspace in question is

$$d^k x \equiv dx_1 \wedge dx_2 \cdots dx_k \quad (5.9)$$

This can be expanded explicitly in coordinates

$$d^k x = da_1 da_2 \cdots da_k \left(\frac{\partial x^{\mu_1}}{\partial a_1} \frac{\partial x^{\mu_2}}{\partial a_2} \cdots \frac{\partial x^{\mu_k}}{\partial a_k} \right) (\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \cdots \wedge \gamma_{\mu_k}) \quad (5.10)$$

Observe that when k is also the dimension of the space, we can employ a pseudoscalar $I = \gamma_0 \gamma_1 \cdots \gamma_k$ and can specify our volume element in terms of the Jacobian determinant.

This is

$$d^k x = I da_1 da_2 \cdots da_k \left| \frac{\partial(x^1, x^2, \cdots, x^k)}{\partial(a_1, a_2, \cdots, a_k)} \right| \quad (5.11)$$

However, we will not have a requirement to express the Stokes result in terms of such Jacobians.

5.2.5 Expansion of the curl and volume element product

We are now prepared to go on to the meat of the issue. The first order of business is the expansion of the curl and volume element product

$$\begin{aligned} (\nabla \wedge F) \cdot d^k x &= (\gamma^\mu \wedge \partial_\mu F) \cdot d^k x \\ &= \langle (\gamma^\mu \wedge \partial_\mu F) d^k x \rangle \end{aligned} \quad (5.12)$$

The wedge product within the scalar grade selection operator can be expanded in symmetric or antisymmetric sums, but this is a grade dependent operation. For odd grade blades A (vector, trivector, ...), and vector a we have for the dot and wedge product respectively

$$\begin{aligned} a \wedge A &= \frac{1}{2}(aA - Aa) \\ a \cdot A &= \frac{1}{2}(aA + Aa) \end{aligned} \quad (5.13)$$

Similarly for even grade blades we have

$$\begin{aligned} a \wedge A &= \frac{1}{2}(aA + Aa) \\ a \cdot A &= \frac{1}{2}(aA - Aa) \end{aligned} \quad (5.14)$$

First treating the odd grade case for F we have

$$(\nabla \wedge F) \cdot d^k x = \frac{1}{2} \langle \gamma^\mu \partial_\mu F d^k x \rangle - \frac{1}{2} \langle \partial_\mu F \gamma^\mu d^k x \rangle \quad (5.15)$$

Employing cyclic scalar reordering within the scalar product for the first term

$$\langle abc \rangle = \langle bca \rangle \quad (5.16)$$

we have

$$\begin{aligned} (\nabla \wedge F) \cdot d^k x &= \frac{1}{2} \langle \partial_\mu F (d^k x \gamma^\mu - \gamma^\mu d^k x) \rangle \\ &= \frac{1}{2} \langle \partial_\mu F (d^k x \cdot \gamma^\mu - \gamma^\mu d^k x) \rangle \\ &= \langle \partial_\mu F (d^k x \cdot \gamma^\mu) \rangle \end{aligned} \quad (5.17)$$

The end result is

$$(\nabla \wedge F) \cdot d^k x = \partial_\mu F \cdot (d^k x \cdot \gamma^\mu) \quad (5.18)$$

For even grade F (and thus odd grade $d^k x$) it is straightforward to show that eq. (5.18) also holds.

5.2.6 Expanding the volume dot product

We want to expand the volume integral dot product

$$d^k x \cdot \gamma^\mu \quad (5.19)$$

Picking $k = 4$ will serve to illustrate the pattern, and the generalization (or degeneralization to lower grades) will be clear. We have

$$\begin{aligned} d^4 x \cdot \gamma^\mu &= (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4) \cdot \gamma^\mu \\ &= (dx_1 \wedge dx_2 \wedge dx_3) dx_4 \cdot \gamma^\mu \\ &\quad - (dx_1 \wedge dx_2 \wedge dx_4) dx_3 \cdot \gamma^\mu \\ &\quad + (dx_1 \wedge dx_3 \wedge dx_4) dx_2 \cdot \gamma^\mu \\ &\quad - (dx_2 \wedge dx_3 \wedge dx_4) dx_1 \cdot \gamma^\mu \end{aligned} \quad (5.20)$$

This avoids the requirement to do the entire Jacobian expansion of eq. (5.11). The dot product of the differential displacement dx_m with γ^μ can now be made explicit without as much mess.

$$\begin{aligned} dx_m \cdot \gamma^\mu &= da_m \frac{\partial x^\nu}{\partial a_m} \gamma_\nu \cdot \gamma^\mu \\ &= da_m \frac{\partial x^\mu}{\partial a_m} \end{aligned} \quad (5.21)$$

We now have products of the form

$$\begin{aligned} \partial_\mu F da_m \frac{\partial x^\mu}{\partial a_m} &= da_m \frac{\partial x^\mu}{\partial a_m} \frac{\partial F}{\partial x^\mu} \\ &= da_m \frac{\partial F}{\partial a_m} \end{aligned} \quad (5.22)$$

Now we see that the differential form of eq. (5.18) for this $k = 4$ example is reduced to

$$\begin{aligned}
(\nabla \wedge F) \cdot d^4x &= da_4 \frac{\partial F}{\partial a_4} \cdot (dx_1 \wedge dx_2 \wedge dx_3) \\
&\quad - da_3 \frac{\partial F}{\partial a_3} \cdot (dx_1 \wedge dx_2 \wedge dx_4) \\
&\quad + da_2 \frac{\partial F}{\partial a_2} \cdot (dx_1 \wedge dx_3 \wedge dx_4) \\
&\quad - da_1 \frac{\partial F}{\partial a_1} \cdot (dx_2 \wedge dx_3 \wedge dx_4)
\end{aligned} \tag{5.23}$$

While eq. (5.18) was a statement of Stokes theorem in this Geometric Algebra formulation, it was really incomplete without this explicit expansion of $(\partial_\mu F) \cdot (d^k x \cdot \gamma^\mu)$. This expansion for the $k = 4$ case serves to illustrate that we would write Stokes theorem as

$$\boxed{\int (\nabla \wedge F) \cdot d^k x = \frac{1}{(k-1)!} \epsilon^{rs\cdots tu} \int da_u \frac{\partial F}{\partial a_u} \cdot (dx_r \wedge dx_s \wedge \cdots \wedge dx_t)} \tag{5.24}$$

Here the indices have the range $\{r, s, \cdots, t, u\} \in \{1, 2, \cdots, k\}$. This with the definitions eq. (5.8), and eq. (5.9) is really Stokes theorem in its full glory.

Observe that in this Geometric algebra form, the one forms $dx_i = da_i \partial x / \partial a_i, i \in [1, k]$ are nothing more abstract than plain old vector differential elements. In the formalism of differential forms, this would be vectors, and $(\nabla \wedge F) \cdot d^k x$ would be a k form. In a context where we are working with vectors, or blades already, the Geometric Algebra statement of the theorem avoids a requirement to translate to the language of forms.

With a statement of the general theorem complete, let us return to our $k = 4$ case where we can now integrate over each of the a_1, a_2, \cdots, a_k parameters. That is

$$\begin{aligned}
\int (\nabla \wedge F) \cdot d^4x &= \int (F(a_4(1)) - F(a_4(0))) \cdot (dx_1 \wedge dx_2 \wedge dx_3) \\
&\quad - \int (F(a_3(1)) - F(a_3(0))) \cdot (dx_1 \wedge dx_2 \wedge dx_4) \\
&\quad + \int (F(a_2(1)) - F(a_2(0))) \cdot (dx_1 \wedge dx_3 \wedge dx_4) \\
&\quad - \int (F(a_1(1)) - F(a_1(0))) \cdot (dx_2 \wedge dx_3 \wedge dx_4)
\end{aligned} \tag{5.25}$$

This is precisely Stokes theorem for the trivector case and makes the enumeration of the boundary surfaces explicit. As derived there was no requirement for an orthonormal basis, nor a Euclidean metric, nor a parametrization along the basis directions. The only requirement of the parametrization is that the associated volume element is non-trivial (i.e. none of $dx_q \wedge dx_r = 0$).

For completeness, note that our boundary surface and associated Stokes statement for the bivector and vector cases is, by inspection respectively

$$\begin{aligned}
\int (\nabla \wedge F) \cdot d^3 x &= \int (F(a_3(1)) - F(a_3(0))) \cdot (dx_1 \wedge dx_2) \\
&\quad - \int (F(a_2(1)) - F(a_2(0))) \cdot (dx_1 \wedge dx_3) \\
&\quad + \int (F(a_1(1)) - F(a_1(0))) \cdot (dx_2 \wedge dx_3)
\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}
\int (\nabla \wedge F) \cdot d^2 x &= \int (F(a_2(1)) - F(a_2(0))) \cdot dx_1 \\
&\quad - \int (F(a_1(1)) - F(a_1(0))) \cdot dx_2
\end{aligned} \tag{5.27}$$

These three expansions can be summarized by the original single statement of eq. (5.1), which repeating for reference, is

$$\int (\nabla \wedge F) \cdot d^k x = \int F \cdot d^{k-1} x \tag{5.28}$$

Where it is implied that the blade F is evaluated on the boundaries and dotted with the associated hypersurface boundary element. However, having expanded this we now have an explicit statement of exactly what that surface element is now for any desired parametrization.

5.3 Duality relations and special cases

Some special (and more recognizable) cases of eq. (5.1) are possible considering specific grades of F , and in some cases employing duality relations.

5.3.1 curl surface integral

One important case is the \mathbb{R}^3 vector result, which can be expressed in terms of the cross product.

Write $\hat{\mathbf{n}} d^2 x = -i dA$. Then we have

$$\begin{aligned}
(\nabla \wedge \mathbf{f}) \cdot d^2 x &= \langle i(\nabla \times \mathbf{f})(-i dA) \rangle \\
&= (\nabla \times \mathbf{f}) \cdot \hat{\mathbf{n}} dA
\end{aligned} \tag{5.29}$$

This recovers the familiar cross product form of Stokes law.

$$\int (\nabla \times \mathbf{f}) \cdot \hat{\mathbf{n}} dA = \oint \mathbf{f} \cdot d\mathbf{x} \tag{5.30}$$

5.3.2 3D divergence theorem

Duality applied to the bivector Stokes result provides the divergence theorem in \mathbb{R}^3 . For bivector B , let $iB = \mathbf{f}$, $d^3 x = i dV$, and $d^2 x = i \hat{\mathbf{n}} dA$. We then have

$$\begin{aligned}
(\nabla \wedge B) \cdot d^3x &= \langle (\nabla \wedge B) \cdot d^3x \rangle \\
&= \frac{1}{2} \langle (\nabla B + B \nabla) i dV \rangle \\
&= \nabla \cdot \mathbf{f} dV
\end{aligned} \tag{5.31}$$

Similarly

$$\begin{aligned}
B \cdot d^2x &= \langle -i \mathbf{f} \hat{\mathbf{n}} dA \rangle \\
&= (\mathbf{f} \cdot \hat{\mathbf{n}}) dA
\end{aligned} \tag{5.32}$$

This recovers the \mathbb{R}^3 divergence equation

$$\int \nabla \cdot \mathbf{f} dV = \int (\mathbf{f} \cdot \hat{\mathbf{n}}) dA \tag{5.33}$$

5.3.3 4D divergence theorem

How about the four dimensional spacetime divergence? Write, express a trivector as a dual four-vector $T = if$, and the four volume element $d^4x = idQ$. This gives

$$\begin{aligned}
(\nabla \wedge T) \cdot d^4x &= \frac{1}{2} \langle (\nabla T - T \nabla) i \rangle dQ \\
&= \frac{1}{2} \langle (\nabla if - if \nabla) i \rangle dQ \\
&= \frac{1}{2} \langle (\nabla f + f \nabla) \rangle dQ \\
&= (\nabla \cdot f) dQ
\end{aligned} \tag{5.34}$$

For the boundary volume integral write $d^3x = nidV$, for

$$\begin{aligned}
T \cdot d^3x &= \langle (if)(ni) \rangle dV \\
&= \langle fn \rangle dV \\
&= (f \cdot n) dV
\end{aligned} \tag{5.35}$$

So we have

$$\int \partial_\mu f^\mu dQ = \int f^\nu n_\nu dV \tag{5.36}$$

the orientation of the fourspace volume element and the boundary normal is defined in terms of the parametrization, the duality relations and our explicit expansion of the 4D stokes boundary integral above.

5.3.4 4D divergence theorem, continued

The basic idea of using duality to express the 4D divergence integral as a stokes boundary surface integral has been explored. Lets consider this in more detail picking a specific parametrization, namely rectangular four vector coordinates. For the volume element write

$$\begin{aligned}
 d^4x &= (\gamma_0 dx^0) \wedge (\gamma_1 dx^1) \wedge (\gamma_2 dx^2) \wedge (\gamma_3 dx^3) \\
 &= \gamma_0 \gamma_1 \gamma_2 \gamma_3 dx^0 dx^1 dx^2 dx^3 \\
 &= i dx^0 dx^1 dx^2 dx^3
 \end{aligned} \tag{5.37}$$

As seen previously (but not separately), the divergence can be expressed as the dual of the curl

$$\begin{aligned}
 \nabla \cdot f &= \langle \nabla f \rangle \\
 &\quad \text{grade 3} \\
 &= - \left\langle \nabla i \left(\overset{\uparrow}{if} \right) \right\rangle \\
 &= \langle i \nabla (if) \rangle \\
 &\quad \text{grade 2} \quad \text{grade 4} \\
 &= \left\langle i \left(\overset{\uparrow}{\nabla \cdot (if)} + \overset{\uparrow}{\nabla \wedge (if)} \right) \right\rangle \\
 &= i \langle \nabla \wedge (if) \rangle
 \end{aligned} \tag{5.38}$$

So we have $\nabla \wedge (if) = -i(\nabla \cdot f)$. Putting things together, and writing $if = -fi$ we have

$$\begin{aligned}
 \int (\nabla \wedge (if)) \cdot d^4x &= \int (\nabla \cdot f) dx^0 dx^1 dx^2 dx^3 \\
 &= \int dx^0 \partial_0 (fi) \cdot \gamma_{123} dx^1 dx^2 dx^3 \\
 &\quad - \int dx^1 \partial_1 (fi) \cdot \gamma_{023} dx^0 dx^2 dx^3 \\
 &\quad + \int dx^2 \partial_2 (fi) \cdot \gamma_{013} dx^0 dx^1 dx^3 \\
 &\quad - \int dx^3 \partial_3 (fi) \cdot \gamma_{012} dx^0 dx^1 dx^2
 \end{aligned} \tag{5.39}$$

It is straightforward to reduce each of these dot products. For example

$$\begin{aligned}
 \partial_2 (fi) \cdot \gamma_{013} &= \langle \partial_2 f \gamma_{0123013} \rangle \\
 &= - \langle \partial_2 f \gamma_2 \rangle \\
 &= - \gamma_2 \partial_2 \cdot f \\
 &= \gamma^2 \partial_2 \cdot f
 \end{aligned} \tag{5.40}$$

The rest proceed the same and rather anticlimactically we end up coming full circle

$$\begin{aligned}
\int (\nabla \cdot f) dx^0 dx^1 dx^2 dx^3 &= \int dx^0 \gamma^0 \partial_0 \cdot f dx^1 dx^2 dx^3 \\
&+ \int dx^1 \gamma^1 \partial_1 \cdot f dx^0 dx^2 dx^3 \\
&+ \int dx^2 \gamma^2 \partial_2 \cdot f dx^0 dx^1 dx^3 \\
&+ \int dx^3 \gamma^3 \partial_3 \cdot f dx^0 dx^1 dx^2
\end{aligned} \tag{5.41}$$

This is however nothing more than the definition of the divergence itself and no need to resort to Stokes theorem is required. However, if we are integrating over a rectangle and perform each of the four integrals, we have (with $c = 1$) from the dual Stokes equation the perhaps less obvious result

$$\begin{aligned}
\int \partial_\mu f^\mu dt dx dy dz &= \int (f^0(t_1) - f^0(t_0)) dx dy dz \\
&+ \int (f^1(x_1) - f^1(x_0)) dt dy dz \\
&+ \int (f^2(y_1) - f^2(y_0)) dt dx dz \\
&+ \int (f^3(z_1) - f^3(z_0)) dt dx dy
\end{aligned} \tag{5.42}$$

When stated this way one sees that this could have just as easily have followed directly from the left hand side. What is the point then of the divergence theorem or Stokes theorem? I think that the value must really be the fact that the Stokes formulation naturally builds the volume element in a fashion independent of any specific parametrization. Here in rectangular coordinates the result seems obvious, but would the equivalent result seem obvious if non-rectangular spacetime coordinates were employed? Probably not.

Exploring Stokes Theorem in tensor form

6.1 Motivation

I have worked through Stokes theorem concepts a couple times on my own now. One of the first times, I was trying to formulate this in a Geometric Algebra context. I had to resort to a tensor decomposition, and pictures, before ending back in the Geometric Algebra description. Later I figured out how to do it entirely with a Geometric Algebra description, and was able to eliminate reliance on the pictures that made the path to generalization to higher dimensional spaces unclear.

It is my expectation that if one started with a tensor description, the proof entirely in tensor form would not be difficult. This is what I had like to try this time. To start off, I will temporarily use the Geometric Algebra curl expression so I know what my tensor equation starting point will be, but once that starting point is found, we can work entirely in coordinate representation. For somebody who already knows that this is the starting point, all of this initial motivation can be skipped.

6.2 Translating the exterior derivative to a coordinate representation

Our starting point is a curl, dotted with a volume element of the same grade, so that the result is a scalar

$$\int d^n x \cdot (\nabla \wedge A). \quad (6.1)$$

Here A is a blade of grade $n - 1$, and we wedge this with the gradient for the space

$$\nabla \equiv e^i \partial_i = e_i \partial^i, \quad (6.2)$$

where we work with a basis (not necessarily orthonormal) $\{e_i\}$, and the reciprocal frame for that basis $\{e^i\}$ defined by the relation

$$e^i \cdot e_j = \delta^i_j. \quad (6.3)$$

Our coordinates in these basis sets are

$$\begin{aligned} x \cdot e^i &\equiv x^i \\ x \cdot e_i &\equiv x_i \end{aligned} \quad (6.4)$$

so that

$$x = x^i e_i = x_i e^i. \quad (6.5)$$

The operator coordinates of the gradient are defined in the usual fashion

$$\begin{aligned} \partial_i &\equiv \frac{\partial}{\partial x^i} \\ \partial^i &\equiv \frac{\partial}{\partial x_i} \end{aligned} \quad (6.6)$$

The volume element for the subspace that we are integrating over we will define in terms of an arbitrary parametrization

$$x = x(\alpha_1, \alpha_2, \dots, \alpha_n) \quad (6.7)$$

The subspace can be considered spanned by the differential elements in each of the respective curves where all but the i th parameter are held constant.

$$dx_{\alpha_i} = d\alpha_i \frac{\partial x}{\partial \alpha_i} = d\alpha_i \frac{\partial x^j}{\partial \alpha_i} e_j. \quad (6.8)$$

We assume that the integral is being performed in a subspace for which none of these differential elements in that region are linearly dependent (i.e. our Jacobean determinant must be non-zero).

The magnitude of the wedge product of all such differential elements provides the volume of the parallelogram, or parallelepiped (or higher dimensional analogue), and is

$$d^n x = d\alpha_1 d\alpha_2 \dots d\alpha_n \frac{\partial x}{\partial \alpha_n} \wedge \dots \wedge \frac{\partial x}{\partial \alpha_2} \wedge \frac{\partial x}{\partial \alpha_1}. \quad (6.9)$$

The volume element is a oriented quantity, and may be adjusted with an arbitrary sign (or equivalently an arbitrary permutation of the differential elements in the wedge product), and we will see that it is convenient for the translation to tensor form, to express these in reversed order.

Let us write

$$d^n \alpha = d\alpha_1 d\alpha_2 \dots d\alpha_n, \quad (6.10)$$

so that our volume element in coordinate form is

$$d^n x = d^n \alpha \frac{\partial x^i}{\partial \alpha_1} \frac{\partial x^j}{\partial \alpha_2} \dots \frac{\partial x^k}{\partial \alpha_{n-1}} \frac{\partial x^l}{\partial \alpha_n} (e_l \wedge e_k \wedge \dots \wedge e_j \wedge e_i). \quad (6.11)$$

Our curl will also be a grade n blade. We write for the $n - 1$ grade blade

$$A = A_{bc\dots d} (e^b \wedge e^c \wedge \dots \wedge e^d), \quad (6.12)$$

where $A_{bc\dots d}$ is antisymmetric (i.e. $A = a_1 \wedge a_2 \wedge \dots \wedge a_{n-1}$ for a some set of vectors $a_i, i \in 1..n - 1$).

With our gradient in coordinate form

$$\nabla = e^a \partial_a, \quad (6.13)$$

the curl is then

$$\nabla \wedge A = \partial_a A_{bc\dots d} (e^a \wedge e^b \wedge e^c \wedge \dots \wedge e^d). \quad (6.14)$$

The differential form for our integral can now be computed by expanding out the dot product. We want

$$(e_l \wedge e_k \wedge \dots \wedge e_j \wedge e_i) \cdot (e^a \wedge e^b \wedge e^c \wedge \dots \wedge e^d) = (((((e_l \wedge e_k \wedge \dots \wedge e_j \wedge e_i) \cdot e^a) \cdot e^b) \cdot e^c) \cdot \dots) \cdot e^d. \quad (6.15)$$

Evaluation of the interior dot products introduces the intrinsic antisymmetry required for Stokes theorem. For example, with

$$\begin{aligned} (e_n \wedge e_{n-1} \wedge \dots \wedge e_2 \wedge e_1) \cdot e^a &= (e_n \wedge e_{n-1} \wedge \dots \wedge e_3 \wedge e_2)(e_1 \cdot e^a) \\ &\quad - (e_n \wedge e_{n-1} \wedge \dots \wedge e_3 \wedge e_1)(e_2 \cdot e^a) \\ &\quad + (e_n \wedge e_{n-1} \wedge \dots \wedge e_2 \wedge e_1)(e_3 \cdot e^a) \\ &\quad \dots \\ &\quad (-1)^{n-1} (e_{n-1} \wedge e_{n-2} \wedge \dots \wedge e_2 \wedge e_1)(e_n \cdot e^a) \end{aligned} \quad (6.16)$$

Since $e_i \cdot e^a = \delta_i^a$ our end result is a completely antisymmetric set of permutations of all the deltas

$$(e_l \wedge e_k \wedge \dots \wedge e_j \wedge e_i) \cdot (e^a \wedge e^b \wedge e^c \wedge \dots \wedge e^d) = \delta^{[a} \delta^b \delta^c \dots \delta^{d]l}, \quad (6.17)$$

and the curl integral takes its coordinate form

$$\int d^n x \cdot (\nabla \wedge A) = \int d^n \alpha \frac{\partial x^i}{\partial \alpha_1} \frac{\partial x^j}{\partial \alpha_2} \dots \frac{\partial x^k}{\partial \alpha_{n-1}} \frac{\partial x^l}{\partial \alpha_n} \partial_a A_{bc\dots d} \delta^{[a} \delta^b \delta^c \dots \delta^{d]l}. \quad (6.18)$$

One final contraction of the paired indices gives us our Stokes integral in its coordinate representation

$$\boxed{\int d^n x \cdot (\nabla \wedge A) = \int d^n \alpha \frac{\partial x^{[a}}{\partial \alpha_1} \frac{\partial x^{b}}{\partial \alpha_2} \dots \frac{\partial x^c}{\partial \alpha_{n-1}} \frac{\partial x^{d]}}{\partial \alpha_n} \partial_a A_{bc\dots d}} \quad (6.19)$$

We now have a starting point that is free of any of the abstraction of Geometric Algebra or differential forms. We can identify the products of partials here as components of a scalar hypervolume element (possibly signed depending on the orientation of the parametrization)

$$d\alpha_1 d\alpha_2 \dots d\alpha_n \frac{\partial x^{[a}}{\partial \alpha_1} \frac{\partial x^{b}}{\partial \alpha_2} \dots \frac{\partial x^c}{\partial \alpha_{n-1}} \frac{\partial x^{d]}}{\partial \alpha_n} \quad (6.20)$$

This is also a specific computation recipe for these hypervolume components, something that may not be obvious when we allow for general metrics for the space. We are also allowing for non-orthonormal coordinate representations, and arbitrary parametrization of the subspace that we are integrating over (our integral need not have the same dimension as the underlying vector space).

Observe that when the number of parameters equals the dimension of the space, we can write out the anti-symmetric term utilizing the determinant of the Jacobian matrix

$$\frac{\partial x^{[a}}{\partial \alpha_1} \frac{\partial x^{b}}{\partial \alpha_2} \dots \frac{\partial x^c}{\partial \alpha_{n-1}} \frac{\partial x^{d]}}{\partial \alpha_n} = \epsilon^{ab\dots d} \left| \frac{\partial(x^1, x^2, \dots, x^n)}{\partial(\alpha_1, \alpha_2, \dots, \alpha_n)} \right| \quad (6.21)$$

When the dimension of the space n is greater than the number of parameters for the integration hypervolume in question, the antisymmetric sum of partials is still the determinant of a Jacobian matrix

$$\frac{\partial x^{[a_1}}{\partial \alpha_1} \frac{\partial x^{a_2}}{\partial \alpha_2} \cdots \frac{\partial x^{a_{n-1}}}{\partial \alpha_{n-1}} \frac{\partial x^{a_n]}{\partial \alpha_n} = \left| \frac{\partial(x^{a_1}, x^{a_2}, \dots, x^{a_n})}{\partial(\alpha_1, \alpha_2, \dots, \alpha_n)} \right|, \quad (6.22)$$

however, we will have one such Jacobian for each unique choice of indices.

6.3 The Stokes work starts here

The task is to relate our integral to the boundary of this volume, coming up with an explicit recipe for the description of that bounding surface, and determining the exact form of the reduced rank integral. This job is essentially to reduce the ranks of the tensors that are being contracted in our Stokes integral. With the derivative applied to our rank $n - 1$ antisymmetric tensor $A_{bc\dots d}$, we can apply the chain rule and examine the permutations so that this can be rewritten as a contraction of A itself with a set of rank $n - 1$ surface area elements.

$$\int d^n \alpha \frac{\partial x^{[a}}{\partial \alpha_1} \frac{\partial x^{b}}{\partial \alpha_2} \cdots \frac{\partial x^{c}}{\partial \alpha_{n-1}} \frac{\partial x^{d]}{\partial \alpha_n} \partial_a A_{bc\dots d} =? \quad (6.23)$$

Now, while the setup here has been completely general, this task is motivated by study of special relativity, where there is a requirement to work in a four dimensional space. Because of that explicit goal, I am not going to attempt to formulate this in a completely abstract fashion. That task is really one of introducing sufficiently general notation. Instead, I am going to proceed with a simpleton approach, and do this explicitly, and repeatedly for each of the rank 1, rank 2, and rank 3 tensor cases. It will be clear how this all generalizes by doing so, should one wish to work in still higher dimensional spaces.

6.3.1 The rank 1 tensor case

The equation we are working with for this vector case is

$$\int d^2 x \cdot (\nabla \wedge A) = \int d\alpha_1 d\alpha_2 \frac{\partial x^{[a}}{\partial \alpha_1} \frac{\partial x^{b]}{\partial \alpha_2} \partial_a A_b(\alpha_1, \alpha_2) \quad (6.24)$$

Expanding out the antisymmetric partials we have

$$\frac{\partial x^{[a}}{\partial \alpha_1} \frac{\partial x^{b]}{\partial \alpha_2} = \frac{\partial x^a}{\partial \alpha_1} \frac{\partial x^b}{\partial \alpha_2} - \frac{\partial x^b}{\partial \alpha_1} \frac{\partial x^a}{\partial \alpha_2}, \quad (6.25)$$

with which we can reduce the integral to

$$\begin{aligned} \int d^2 x \cdot (\nabla \wedge A) &= \int \left(d\alpha_1 \frac{\partial x^a}{\partial \alpha_1} \frac{\partial A_b}{\partial x^a} \right) \frac{\partial x^b}{\partial \alpha_2} d\alpha_2 - \left(d\alpha_2 \frac{\partial x^a}{\partial \alpha_2} \frac{\partial A_b}{\partial x^a} \right) \frac{\partial x^b}{\partial \alpha_1} d\alpha_1 \\ &= \int \left(d\alpha_1 \frac{\partial A_b}{\partial \alpha_1} \right) \frac{\partial x^b}{\partial \alpha_2} d\alpha_2 - \left(d\alpha_2 \frac{\partial A_b}{\partial \alpha_2} \right) \frac{\partial x^b}{\partial \alpha_1} d\alpha_1 \end{aligned} \quad (6.26)$$

Now, if it happens that

$$\frac{\partial}{\partial \alpha_1} \frac{\partial x^a}{\partial \alpha_2} = \frac{\partial}{\partial \alpha_2} \frac{\partial x^a}{\partial \alpha_1} = 0 \quad (6.27)$$

then each of the individual integrals in $d\alpha_1$ and $d\alpha_2$ can be carried out. In that case, without any real loss of generality we can designate the integration bounds over the unit parametrization space square $\alpha_i \in [0, 1]$, allowing this integral to be expressed as

$$\begin{aligned} & \int d\alpha_1 d\alpha_2 \frac{\partial x^{[a}}{\partial \alpha_1} \frac{\partial x^{b]}}{\partial \alpha_2} \partial_a A_b(\alpha_1, \alpha_2) \\ &= \int (A_b(1, \alpha_2) - A_b(0, \alpha_2)) \frac{\partial x^b}{\partial \alpha_2} d\alpha_2 - (A_b(\alpha_1, 1) - A_b(\alpha_1, 0)) \frac{\partial x^b}{\partial \alpha_1} d\alpha_1. \end{aligned} \quad (6.28)$$

It is also fairly common to see $A|_{\partial\alpha_i}$ used to designate evaluation of this first integral on the boundary, and using this we write

$$\int d\alpha_1 d\alpha_2 \frac{\partial x^{[a}}{\partial \alpha_1} \frac{\partial x^{b]}}{\partial \alpha_2} \partial_a A_b(\alpha_1, \alpha_2) = \int A_b|_{\partial\alpha_1} \frac{\partial x^b}{\partial \alpha_2} d\alpha_2 - A_b|_{\partial\alpha_2} \frac{\partial x^b}{\partial \alpha_1} d\alpha_1. \quad (6.29)$$

Also note that since we are summing over all a, b , and have

$$\frac{\partial x^{[a}}{\partial \alpha_1} \frac{\partial x^{b]}}{\partial \alpha_2} = -\frac{\partial x^{[b}}{\partial \alpha_1} \frac{\partial x^{a]}}{\partial \alpha_2}, \quad (6.30)$$

we can write this summing over all unique pairs of a, b instead, which eliminates a small bit of redundancy (especially once the dimension of the vector space gets higher)

$$\boxed{\sum_{a < b} \int d\alpha_1 d\alpha_2 \frac{\partial x^{[a}}{\partial \alpha_1} \frac{\partial x^{b]}}{\partial \alpha_2} (\partial_a A_b - \partial_b A_a) = \int A_b|_{\partial\alpha_1} \frac{\partial x^b}{\partial \alpha_2} d\alpha_2 - A_b|_{\partial\alpha_2} \frac{\partial x^b}{\partial \alpha_1} d\alpha_1.} \quad (6.31)$$

In this form we have recovered the original geometric structure, with components of the curl multiplied by the component of the area element that shares the orientation and direction of that portion of the curl bivector.

This form of the result with evaluation at the boundaries in this form, assumed that $\partial x^a / \partial \alpha_1$ was not a function of α_2 and $\partial x^a / \partial \alpha_2$ was not a function of α_1 . When that is not the case, we appear to have a less pretty result

$$\boxed{\sum_{a < b} \int d\alpha_1 d\alpha_2 \frac{\partial x^{[a}}{\partial \alpha_1} \frac{\partial x^{b]}}{\partial \alpha_2} (\partial_a A_b - \partial_b A_a) = \int d\alpha_2 \int d\alpha_1 \frac{\partial A_b}{\partial \alpha_1} \frac{\partial x^b}{\partial \alpha_2} - \int d\alpha_2 \int d\alpha_1 \frac{\partial A_b}{\partial \alpha_2} \frac{\partial x^b}{\partial \alpha_1}} \quad (6.32)$$

Can this be reduced any further in the general case? Having seen the statements of Stokes theorem in its differential forms formulation, I initially expected the answer was yes, and only when I got to evaluating my \mathbb{R}^4 spacetime example below did I realize that the differential displacements for the parallelogram that constituted the area element were functions of both parameters. Perhaps this detail is there in the differential forms version of the general Stokes theorem too, but is just hidden in a tricky fashion by the compact notation.

Sanity check: \mathbb{R}^2 case in rectangular coordinates

For $x^1 = x, x^2 = y$, and $\alpha_1 = x, \alpha_2 = y$, we have for the LHS

$$\begin{aligned} & \int_{x=x_0}^{x_1} \int_{y=y_0}^{y_1} dxdy \left(\frac{\partial x^1}{\partial \alpha_1} \frac{\partial x^2}{\partial \alpha_2} - \frac{\partial x^2}{\partial \alpha_1} \frac{\partial x^1}{\partial \alpha_2} \right) \partial_1 A_2 + \left(\frac{\partial x^2}{\partial \alpha_1} \frac{\partial x^1}{\partial \alpha_2} - \frac{\partial x^1}{\partial \alpha_1} \frac{\partial x^2}{\partial \alpha_2} \right) \partial_2 A_1 \\ &= \int_{x=x_0}^{x_1} \int_{y=y_0}^{y_1} dxdy \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \end{aligned} \quad (6.33)$$

Our RHS expands to

$$\begin{aligned} & \int_{y=y_0}^{y_1} dy \left((A_1(x_1, y) - A_1(x_0, y)) \frac{\partial x^1}{\partial y} + (A_2(x_1, y) - A_2(x_0, y)) \frac{\partial x^2}{\partial y} \right) \\ & \quad - \int_{x=x_0}^{x_1} dx \left((A_1(x, y_1) - A_1(x, y_0)) \frac{\partial x^1}{\partial x} + (A_2(x, y_1) - A_2(x, y_0)) \frac{\partial x^2}{\partial x} \right) \\ &= \int_{y=y_0}^{y_1} dy (A_y(x_1, y) - A_y(x_0, y)) - \int_{x=x_0}^{x_1} dx (A_x(x, y_1) - A_x(x, y_0)) \end{aligned} \quad (6.34)$$

We have

$$\begin{aligned} & \int_{x=x_0}^{x_1} \int_{y=y_0}^{y_1} dxdy \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\ &= \int_{y=y_0}^{y_1} dy (A_y(x_1, y) - A_y(x_0, y)) - \int_{x=x_0}^{x_1} dx (A_x(x, y_1) - A_x(x, y_0)) \end{aligned} \quad (6.35)$$

The RHS is just a positively oriented line integral around the rectangle of integration

$$\int A_x(x, y_0) \hat{\mathbf{x}} \cdot (\hat{\mathbf{x}} dx) + A_y(x_1, y) \hat{\mathbf{y}} \cdot (\hat{\mathbf{y}} dy) + A_x(x, y_1) \hat{\mathbf{x}} \cdot (-\hat{\mathbf{x}} dx) + A_y(x_0, y) \hat{\mathbf{y}} \cdot (-\hat{\mathbf{y}} dy) = \oint \mathbf{A} \cdot d\mathbf{r}. \quad (6.36)$$

This special case is also recognizable as Green's theorem, evident with the substitution $A_x = P, A_y = Q$, which gives us

$$\int_A dxdy \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \oint_C P dx + Q dy. \quad (6.37)$$

Strictly speaking, Green's theorem is more general, since it applies to integration regions more general than rectangles, but that generalization can be arrived at easily enough, once the region is broken down into adjoining elementary regions.

Sanity check: \mathbb{R}^3 case in rectangular coordinates

It is expected that we can recover the classical Kelvin-Stokes theorem if we use rectangular coordinates in \mathbb{R}^3 . However, we see that we have to consider three different parametrizations. If one picks rectangular parametrizations $(\alpha_1, \alpha_2) = \{(x, y), (y, z), (z, x)\}$ in sequence, in each case holding the value of the additional coordinate fixed, we get three different independent Green's function like relations

$$\begin{aligned}
\int_A dx dy \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) &= \oint_C A_x dx + A_y dy \\
\int_A dy dz \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) &= \oint_C A_y dy + A_z dz \\
\int_A dz dx \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) &= \oint_C A_z dz + A_x dx.
\end{aligned} \tag{6.38}$$

Note that we cannot just add these to form a complete integral $\oint \mathbf{A} \cdot d\mathbf{r}$ since the curves are all have different orientations. To recover the \mathbb{R}^3 Stokes theorem in rectangular coordinates, it appears that we would have to consider a Riemann sum of triangular surface elements, and relate that to the loops over each of the surface elements. In that limiting argument, only the boundary of the complete surface would contribute to the RHS of the relation.

All that said, we should not actually have to go to all this work. Instead we can stick to a two variable parametrization of the surface, and use eq. (6.31) directly.

An illustration for a \mathbb{R}^4 spacetime surface

Suppose we have a particle trajectory defined by an active Lorentz transformation from an initial spacetime point

$$x^i = O^{ij} x_j(0) = O^{ij} g_{jk} x^k = O^i_k x^k(0) \tag{6.39}$$

Let the Lorentz transformation be formed by a composition of boost and rotation

$$\begin{aligned}
O^i_j &= L^i_k R^k_j \\
L^i_j &= \begin{bmatrix} \cosh \alpha & -\sinh \alpha & 0 & 0 \\ -\sinh \alpha & \cosh \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
R^i_j &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \cos \alpha & \sin \alpha & 0 & 0 \\ -\sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{6.40}$$

Different rates of evolution of α and θ define different trajectories, and taken together we have a surface described by the two parameters

$$x^i(\alpha, \theta) = L^i_k R^k_j x^j(0, 0). \tag{6.41}$$

We can compute displacements along the trajectories formed by keeping either α or θ fixed and varying the other. Those are

$$\begin{aligned}\frac{\partial x^i}{\partial \alpha} d\alpha &= \frac{dL^i_k}{d\alpha} R^k_j x^j(0,0) \\ \frac{\partial x^i}{\partial \theta} d\theta &= L^i_k \frac{dR^k_j}{d\theta} x^j(0,0).\end{aligned}\tag{6.42}$$

Writing $y^i = x^i(0,0)$ the computation of the partials above yields

$$\begin{aligned}\frac{\partial x^i}{\partial \alpha} &= \begin{bmatrix} \sinh \alpha y^0 - \cosh \alpha (\cos \theta y^1 + \sin \theta y^2) \\ -\cosh \alpha y^0 + \sinh \alpha (\cos \theta y^1 + \sin \theta y^2) \\ 0 \\ 0 \end{bmatrix} \\ \frac{\partial x^i}{\partial \theta} &= \begin{bmatrix} -\sinh \alpha (-\sin \theta y^1 + \cos \theta y^2) \\ \cosh \alpha (-\sin \theta y^1 + \cos \theta y^2) \\ -(\cos \theta y^1 + \sin \theta y^2) \\ 0 \end{bmatrix}\end{aligned}\tag{6.43}$$

Different choices of the initial point y^i yield different surfaces, but we can get the idea by picking a simple starting point $y^i = (0, 1, 0, 0)$ leaving

$$\begin{aligned}\frac{\partial x^i}{\partial \alpha} &= \begin{bmatrix} -\cosh \alpha \cos \theta \\ \sinh \alpha \cos \theta \\ 0 \\ 0 \end{bmatrix} \\ \frac{\partial x^i}{\partial \theta} &= \begin{bmatrix} \sinh \alpha \sin \theta \\ -\cosh \alpha \sin \theta \\ -\cos \theta \\ 0 \end{bmatrix}.\end{aligned}\tag{6.44}$$

We can now compute our Jacobian determinants

$$\frac{\partial x^{[a}}{\partial \alpha} \frac{\partial x^{b]}{\partial \theta} = \left| \frac{\partial(x^a, x^b)}{\partial(\alpha, \theta)} \right|.\tag{6.45}$$

Those are

$$\begin{aligned}
\left| \frac{\partial(x^0, x^1)}{\partial(\alpha, \theta)} \right| &= \cos \theta \sin \theta \\
\left| \frac{\partial(x^0, x^2)}{\partial(\alpha, \theta)} \right| &= \cosh \alpha \cos^2 \theta \\
\left| \frac{\partial(x^0, x^3)}{\partial(\alpha, \theta)} \right| &= 0 \\
\left| \frac{\partial(x^1, x^2)}{\partial(\alpha, \theta)} \right| &= -\sinh \alpha \cos^2 \theta \\
\left| \frac{\partial(x^1, x^3)}{\partial(\alpha, \theta)} \right| &= 0 \\
\left| \frac{\partial(x^2, x^3)}{\partial(\alpha, \theta)} \right| &= 0
\end{aligned} \tag{6.46}$$

Using this, let us see a specific 4D example in spacetime for the integral of the curl of some four vector A^i , enumerating all the non-zero components of eq. (6.32) for this particular spacetime surface

$$\sum_{a < b} \int d\alpha d\theta \left| \frac{\partial(x^a, x^b)}{\partial(\alpha, \theta)} \right| (\partial_a A_b - \partial_b A_a) = \int d\theta \int d\alpha \frac{\partial A_b}{\partial \alpha} \frac{\partial x^b}{\partial \theta} - \int d\theta \int d\alpha \frac{\partial A_b}{\partial \theta} \frac{\partial x^b}{\partial \alpha} \tag{6.47}$$

The LHS is thus found to be

$$\begin{aligned}
&\int d\alpha d\theta \left(\left| \frac{\partial(x^0, x^1)}{\partial(\alpha, \theta)} \right| (\partial_0 A_1 - \partial_1 A_0) + \left| \frac{\partial(x^0, x^2)}{\partial(\alpha, \theta)} \right| (\partial_0 A_2 - \partial_2 A_0) + \left| \frac{\partial(x^1, x^2)}{\partial(\alpha, \theta)} \right| (\partial_1 A_2 - \partial_2 A_1) \right) \\
&= \int d\alpha d\theta \left(\cos \theta \sin \theta (\partial_0 A_1 - \partial_1 A_0) + \cosh \alpha \cos^2 \theta (\partial_0 A_2 - \partial_2 A_0) - \sinh \alpha \cos^2 \theta (\partial_1 A_2 - \partial_2 A_1) \right)
\end{aligned} \tag{6.48}$$

On the RHS we have

$$\begin{aligned}
&\int d\theta \int d\alpha \frac{\partial A_b}{\partial \alpha} \frac{\partial x^b}{\partial \theta} - \int d\theta \int d\alpha \frac{\partial A_b}{\partial \theta} \frac{\partial x^b}{\partial \alpha} \\
&= \int d\theta \int d\alpha \left[\sinh \alpha \sin \theta \quad -\cosh \alpha \sin \theta \quad -\cos \theta \quad 0 \right] \frac{\partial}{\partial \alpha} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} \\
&\quad - \int d\theta \int d\alpha \left[-\cosh \alpha \cos \theta \quad \sinh \alpha \cos \theta \quad 0 \quad 0 \right] \frac{\partial}{\partial \theta} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix}
\end{aligned} \tag{6.49}$$

$$\begin{aligned}
& \int d\alpha d\theta \cos \theta \sin \theta (\partial_0 A_1 - \partial_1 A_0) \\
& + \int d\alpha d\theta \cosh \alpha \cos^2 \theta (\partial_0 A_2 - \partial_2 A_0) \\
& - \int d\alpha d\theta \sinh \alpha \cos^2 \theta (\partial_1 A_2 - \partial_2 A_1) \\
& = \int d\theta \sin \theta \int d\alpha \left(\sinh \alpha \frac{\partial A_0}{\partial \alpha} - \cosh \alpha \frac{\partial A_1}{\partial \alpha} \right) \\
& - \int d\theta \cos \theta \int d\alpha \frac{\partial A_2}{\partial \alpha} \\
& + \int d\alpha \cosh \alpha \int d\theta \cos \theta \frac{\partial A_0}{\partial \theta} \\
& - \int d\alpha \sinh \alpha \int d\theta \cos \theta \frac{\partial A_1}{\partial \theta}
\end{aligned} \tag{6.50}$$

Because of the complexity of the surface, only the second term on the RHS has the “evaluate on the boundary” characteristic that may have been expected from a Green’s theorem like line integral.

It is also worthwhile to point out that we have had to be very careful with upper and lower indices all along (and have done so with the expectation that our application would include the special relativity case where our metric determinant is minus one.) Because we worked with upper indices for the area element, we had to work with lower indices for the four vector and the components of the gradient that we included in our curl evaluation.

6.3.2 The rank 2 tensor case

Let us consider briefly the terms in the contraction sum

$$\left| \frac{\partial(x^a, x^b, x^c)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| \partial_a A_{bc} \tag{6.51}$$

For any choice of a set of three distinct indices $(a, b, c) \in \{(0, 1, 2), (0, 1, 3), (0, 2, 3), (1, 2, 3)\}$, we have $6 = 3!$ ways of permuting those indices in this sum

$$\begin{aligned}
\left| \frac{\partial(x^a, x^b, x^c)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| \partial_a A_{bc} &= \sum_{a < b < c} \left| \frac{\partial(x^a, x^b, x^c)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| \partial_a A_{bc} + \left| \frac{\partial(x^a, x^c, x^b)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| \partial_a A_{cb} + \left| \frac{\partial(x^b, x^c, x^a)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| \partial_b A_{ca} \\
&+ \left| \frac{\partial(x^b, x^a, x^c)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| \partial_b A_{ac} + \left| \frac{\partial(x^c, x^a, x^b)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| \partial_c A_{ab} + \left| \frac{\partial(x^c, x^b, x^a)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| \partial_c A_{ba} \\
&= 2! \sum_{a < b < c} \left| \frac{\partial(x^a, x^b, x^c)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| (\partial_a A_{bc} + \partial_b A_{ca} + \partial_c A_{ab})
\end{aligned} \tag{6.52}$$

Observe that we have no sign alternation like we had in the vector (rank 1 tensor) case. That sign alternation in this summation expansion appears to occur only for odd grade tensors.

Returning to the problem, we wish to expand the determinant in order to apply a chain rule contraction as done in the rank-1 case. This can be done along any of rows or columns of the determinant, and we can write any of

$$\begin{aligned}
\left| \frac{\partial(x^a, x^b, x^c)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| &= \frac{\partial x^a}{\partial \alpha_1} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_2, \alpha_3)} \right| - \frac{\partial x^a}{\partial \alpha_2} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_1, \alpha_3)} \right| + \frac{\partial x^a}{\partial \alpha_3} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_1, \alpha_2)} \right| \\
&= \frac{\partial x^b}{\partial \alpha_1} \left| \frac{\partial(x^c, x^a)}{\partial(\alpha_2, \alpha_3)} \right| - \frac{\partial x^b}{\partial \alpha_2} \left| \frac{\partial(x^c, x^a)}{\partial(\alpha_1, \alpha_3)} \right| + \frac{\partial x^b}{\partial \alpha_3} \left| \frac{\partial(x^c, x^a)}{\partial(\alpha_1, \alpha_2)} \right| \\
&= \frac{\partial x^c}{\partial \alpha_1} \left| \frac{\partial(x^a, x^b)}{\partial(\alpha_2, \alpha_3)} \right| - \frac{\partial x^c}{\partial \alpha_2} \left| \frac{\partial(x^a, x^b)}{\partial(\alpha_1, \alpha_3)} \right| + \frac{\partial x^c}{\partial \alpha_3} \left| \frac{\partial(x^a, x^b)}{\partial(\alpha_1, \alpha_2)} \right|
\end{aligned} \tag{6.53}$$

This allows the contraction of the index a , eliminating it from the result

$$\begin{aligned}
\left| \frac{\partial(x^a, x^b, x^c)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| \partial_a A_{bc} &= \left(\frac{\partial x^a}{\partial \alpha_1} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_2, \alpha_3)} \right| - \frac{\partial x^a}{\partial \alpha_2} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_1, \alpha_3)} \right| + \frac{\partial x^a}{\partial \alpha_3} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_1, \alpha_2)} \right| \right) \frac{\partial A_{bc}}{\partial x^a} \\
&= \frac{\partial A_{bc}}{\partial \alpha_1} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_2, \alpha_3)} \right| - \frac{\partial A_{bc}}{\partial \alpha_2} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_1, \alpha_3)} \right| + \frac{\partial A_{bc}}{\partial \alpha_3} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_1, \alpha_2)} \right| \\
&= 2! \sum_{b < c} \frac{\partial A_{bc}}{\partial \alpha_1} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_2, \alpha_3)} \right| - \frac{\partial A_{bc}}{\partial \alpha_2} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_1, \alpha_3)} \right| + \frac{\partial A_{bc}}{\partial \alpha_3} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_1, \alpha_2)} \right|
\end{aligned} \tag{6.54}$$

Dividing out the common $2!$ terms, we can summarize this result as

$$\begin{aligned}
&\sum_{a < b < c} \int d\alpha_1 d\alpha_2 d\alpha_3 \left| \frac{\partial(x^a, x^b, x^c)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right| (\partial_a A_{bc} + \partial_b A_{ca} + \partial_c A_{ab}) \\
&= \sum_{b < c} \int d\alpha_2 d\alpha_3 \int d\alpha_1 \frac{\partial A_{bc}}{\partial \alpha_1} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_2, \alpha_3)} \right| \\
&- \sum_{b < c} \int d\alpha_1 d\alpha_3 \int d\alpha_2 \frac{\partial A_{bc}}{\partial \alpha_2} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_1, \alpha_3)} \right| \\
&+ \sum_{b < c} \int d\alpha_1 d\alpha_2 \int d\alpha_3 \frac{\partial A_{bc}}{\partial \alpha_3} \left| \frac{\partial(x^b, x^c)}{\partial(\alpha_1, \alpha_2)} \right|
\end{aligned} \tag{6.55}$$

In general, as observed in the spacetime surface example above, the two index Jacobians can be functions of the integration variable first being eliminated. In the special cases where this is not the case (such as the \mathbb{R}^3 case with rectangular coordinates), then we are left with just the evaluation of the tensor element A_{bc} on the boundaries of the respective integrals.

6.3.3 The rank 3 tensor case

The key step is once again just a determinant expansion

$$\begin{aligned}
& \left| \frac{\partial(x^a, x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} \right| \\
&= \frac{\partial x^a}{\partial \alpha_1} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_2, \alpha_3, \alpha_4)} \right| - \frac{\partial x^a}{\partial \alpha_2} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_3, \alpha_4)} \right| + \frac{\partial x^a}{\partial \alpha_3} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_2, \alpha_4)} \right| + \frac{\partial x^a}{\partial \alpha_4} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right|
\end{aligned} \tag{6.56}$$

so that the sum can be reduced from a four index contraction to a 3 index contraction

$$\begin{aligned}
& \left| \frac{\partial(x^a, x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} \right| \partial_a A_{bcd} \\
&= \frac{\partial A_{bcd}}{\partial \alpha_1} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_2, \alpha_3, \alpha_4)} \right| - \frac{\partial A_{bcd}}{\partial \alpha_2} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_3, \alpha_4)} \right| + \frac{\partial A_{bcd}}{\partial \alpha_3} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_2, \alpha_4)} \right| + \frac{\partial A_{bcd}}{\partial \alpha_4} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right|
\end{aligned} \tag{6.57}$$

That is the essence of the theorem, but we can play the same combinatorial reduction games to reduce the built in redundancy in the result

$$\begin{aligned}
& \frac{1}{3!} \int d^4 \alpha \left| \frac{\partial(x^a, x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} \right| \partial_a A_{bcd} \\
&= \sum_{a < b < c < d} \int d^4 \alpha \left| \frac{\partial(x^a, x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} \right| (\partial_a A_{bcd} - \partial_b A_{cda} + \partial_c A_{dab} - \partial_d A_{abc}) \\
&= \sum_{b < c < d} \int d\alpha_2 d\alpha_3 d\alpha_4 \int d\alpha_1 \frac{\partial A_{bcd}}{\partial \alpha_1} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_2, \alpha_3, \alpha_4)} \right| \\
&\quad - \sum_{b < c < d} \int d\alpha_1 d\alpha_3 d\alpha_4 \int d\alpha_2 \frac{\partial A_{bcd}}{\partial \alpha_2} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_3, \alpha_4)} \right| \\
&\quad + \sum_{b < c < d} \int d\alpha_1 d\alpha_2 d\alpha_4 \int d\alpha_3 \frac{\partial A_{bcd}}{\partial \alpha_3} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_2, \alpha_4)} \right| \\
&\quad + \sum_{b < c < d} \int d\alpha_1 d\alpha_2 d\alpha_3 \int d\alpha_4 \frac{\partial A_{bcd}}{\partial \alpha_4} \left| \frac{\partial(x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \right|
\end{aligned} \tag{6.58}$$

6.3.4 A note on Four divergence

Our four divergence integral has the following form

$$\int d^4 \alpha \left| \frac{\partial(x^1, x^2, x^3, x^4)}{\partial(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} \right| \partial_a A^a \tag{6.59}$$

We can relate this to the rank 3 Stokes theorem with a duality transformation, multiplying with a pseudoscalar

$$A^a = \epsilon^{abcd} T_{bcd}, \tag{6.60}$$

where T_{bcd} can also be related back to the vector by the same sort of duality transformation

$$A^a \epsilon_{abcd} = \epsilon^{abcd} \epsilon_{abcd} T_{bcd} = 4! T_{bcd}. \quad (6.61)$$

The divergence integral in terms of the rank 3 tensor is

$$\int d^4\alpha \left| \frac{\partial(x^1, x^2, x^2, x^4)}{\partial(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} \right| \partial_a \epsilon^{abcd} T_{bcd} = \int d^4\alpha \left| \frac{\partial(x^a, x^b, x^c, x^d)}{\partial(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} \right| \partial_a T_{bcd}, \quad (6.62)$$

and we are free to perform the same Stokes reduction of the integral. Of course, this is particularly simple in rectangular coordinates. I still have to think though one subtlty that I feel may be important. We could have started off with an integral of the following form

$$\int dx^1 dx^2 dx^3 dx^4 \partial_a A^a, \quad (6.63)$$

and I think this differs from our starting point slightly because this has none of the antisymmetric structure of the signed 4 volume element that we have used. We do not take the absolute value of our Jacobians anywhere.

Bibliography

- [1] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, Cambridge, UK, 1st edition, 2003.